Overlapping Generations Models

with Realistic Demography:

Statics and Dynamics.

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1 Introduction

Samuelson (1958), with a very simple demo-economic model, was able to raise such fundamental questions as whether a market economy could reach an optimal equilibrium. His extraordinarily rich analysis included both static and dynamic aspects. First, he described different steady states consistent with certain economic constraints. Second, he considered which of these steady states might or might not be reached by a market economy.

During the nearly four decades since his article was published, additional results on these topics have been established. Starret (1972), Gale (1973), and Willis (1988) developed some results for comparative steady states in the discrete N age-group case. Many static results have been established for the two age group model (Diamond, 1965; Gale (1972); Balasko, Cass and Shell (1980); Balasko and Shell (1980, 1981a and 1981b); and Esteban, Mitra and Ray (1993).

By contrast, there have been few studies of the dynamics of overlapping generations models, which is a much more complex topic. Gale (1973) made important progress with the two age-group pure exchange economy, and conjectured about the N age-group case. A number of results have been obtained for the two age-group productive economy as well, as in Gale (1972), Tirole (1985), Weil (1987), Galor and Ryder (1989, 1991) and Galor (1992).

All these works are based on very simplistic demographic assumptions, with the population divided into a finite number of age groups, typically only two, and everybody dying at the end of the last age group. Results often depend strongly on such assumptions; in Samuelson's article, for example, the three age-group model can support steady states which are impossible in the two age-group model. Two age group models are not capable of representing the most basic feature of the human economic life cycle: that it begins and ends with periods of dependency, separated by a long intermediate period of consuming more than is produced. Models in which all survive to the end of the last age group cannot be used to investigate the consequences of mortality change.

Some researchers have extended Samuelson's steady state analysis to the demographically realistic case of continuous age and arbitrary mortality in a productive economy (Arthur and McNicoll, 1978; Lee, 1980, 1994a and 1994b). These more realistic demographic models avoid many of the special assumptions of the literature reviewed above, and permit exploration of important issues such as population aging and the consequences of mortality decline. They also lend themselves more readily to empirical implementation (Lee, 1994a and 1994b).

In this article we consider a continuous demographic model, with a general mortality pattern, and study both static and dynamic properties of a productive market economy. Most of the static results known for the N age-group models are extended to the continuous model. Some results, previously established for economies without capital, will be extended to productive economies. Our work also makes important progress on the dynamic properties. Almost nothing was known for models with more than two age-groups, even for the simplest N age-group model. In the most general case of our continuous model with non-trivial mortality we are able to obtain some results about the stability of some steady-states. In particular, part of Gale's conjecture is proved. Also, at the end of this article, we will discuss non-competitive economies with government taxes or with intergenerational transfers, such as a Social Security System or bequests.

The results are based on a theoretical framework which allows us to get most results in quite a simple way. Surprisingly enough, although we relax some crucial hypotheses, our proofs are often shorter than those presented previously. Moreover, while all our economic statements suppose that the population is in steady-state, the main result of our theoretical approach is obtained without this assumption. It therefore opens the door for more progress in the study of the non-stable population cases.

The remainder of the article is organized as follows. In Section 2 we present our theoretical accounting framework. In Section 3 we first set up the assumptions for our market economy model. Then two subsections study the static and dynamic properties, respectively. Section 4 will examine the case of more general economies where there are some intergenerational transfers or government taxes. The main technical proofs are in an appendix.

2 Theoretical background

In this section we develop a general theoretical accounting framework which will be very useful for the next sections. At this point of the study, rather than giving a precise name to the object we consider (like "consumption", "income", "transfers" or "savings") we will speak about a generic object that we call a "system of reallocation". A system of reallocation g is a function $g(x,t)$ of age x and time t which says that at time t the expected net in-flow (or out-flow, depending on the convention) from the system g for the average individual of age x is $g(x,t)$.[1] For example if g is savings, then $g(x,t)$ equals the net savings at time t by individuals of age x. If g is Social Security then $g(x,t)$ is the difference between benefits received and taxes paid by people of age x at time t. We will suppose that the population is homogeneous in the sense that all individuals of the same cohort have the same expectations although most results hold for heterogeneous population as well[2]. We will define some concepts relative to any system of reallocation, such as the "competitiveness", the "conservativeness" or the wealth held through a system of reallocation, and see how these concepts are dynamically related. While it may seem very abstract at first glance and quite distant from our main objective, the economic meaning

[1] We will refer to individuals throughout, but the framework and results can equally be applied to households by age of head. The issues are discussed in Lee (1980) and (1994b). Alternatively, we could refer to individuals, but suppose the economic life to begin with adulthood, say at age 20.

[2] All our results, except Propositions 3.3 and 3.4, and Theorems 3.5, 3.6 and 3.7, hold for heterogeneous populations if we assume that demographic and economic heterogeneity are independent. All the results, without exception, hold if, in addition, we suppose that the utility function is homothetic.

and the interest of dealing with such notions will become clearer in the next sections.

Throughout the present section we consider very general demographic hypotheses. The population is not assumed to be stable and we call $p(x,t)$ the survival pattern of individuals. By this we mean that $p(x,t)$ is the probability that an individual born x years ago (at time $t-x$) is still alive at time t. We suppose that $p(x,t)$ is integrable and that there exists a "maximum age" ω such that $p(x,t)=0$ for all $x \geq \omega$. The number of births at time t is noted $B(t)$ and the size of the population is $P(t)$. The population is assumed closed to migrations. We assume that the rate of interest is an integrable function of time, $r(t)$.

2.1 Some relevant characteristics of the systems of reallocation.

Viewing a reallocation system from a life cycle or longitudinal perspective we define the present value of expectation of net receipt at birth by :

$$\mathrm{PV}(g,t) = \int_0^\omega e^{-\int_t^{t+x} r(a)da} p(x,t+x)g(x,t+x)dx$$

Viewing a reallocation system from a cross-sectional perspective, at a given time t, we define the population-weighted average flow by :

$$\mathrm{Pop}(g,t) = \frac{1}{P(t)} \int_0^\omega B(t-x)p(x,t)g(x,t)dx$$

(note that the number of people of age x at time t is $B(t-x)p(x,t)$.)

PV and Pop are linear applications whose kernels are sub-spaces which have straightforward economic meanings :

Definition 2.1 *We will say that a system of reallocation, g ,is :*

- competitive if $PV(g,t)=0$ for every t.

- *conservative if $Pop(g,t) = 0$ for every t.*

The "competitiveness" of a system of reallocation indicates its neutrality, in the sense that to add a competitive system of reallocation does not change the present value of the expectation at birth of the average agent. The "conservativeness" indicates whether or not the transactions of a system of reallocation aggregate to zero at any time. For example, in the set of the competitive systems of reallocation we find any kind of competitive savings (for agents with perfect foresight), while the set of conservative systems includes all kinds of Pay-As-You-Go inter-generational transfers and familial transfers such as bequests. In general, some of the competitive systems are not conservative (such as investment), while some conservative systems are not competitive (such as a Pay-As-You-Go pension system). Credit market transactions are both competitive and conservative.

This nomenclature helps formalize different kinds of economic assumptions that are usually made. Throughout this article we denote by $c(x,t)$ and $y_l(x,t)$ the consumption and labor income of individuals of age x at time t. A pure exchange market economy, as considered by Samuelson (1958) and Gale (1973) is characterized by the fact that there is no durable good and, therefore, no possible accumulation, which means that $\mathrm{Pop}(c,t)$ must equal $\mathrm{Pop}(y_l,t)$, and that individual savings are competitive, which says that $\mathrm{PV}(c,t) = \mathrm{PV}(y_l,t)$ at any time. If we are in a market economy with capital then the former condition does not need to hold anymore. Indeed, in a closed market economy the condition $\mathrm{Pop}(c,t) = \mathrm{Pop}(y_l,t)$ is replaced by $P(t)\mathrm{Pop}(y_l - c, t) = \frac{dK}{dt}(t) - r(t)K(t)$ where $K(t)$ is the aggregate capital at time t [3]. Such economies will be studied in Section 3 of this article. Situations where $\mathrm{PV}(c,t)$ and $\mathrm{PV}(y_l,t)$ are not always equal are characteristic of non-competitive economies, such as economies where

[3] For simplicity, we assume throughout this article that there is no depreciation of the capital. Generalization to an economy where the capital depreciates uniformly at a rate $\lambda(t)$ would be straightforward. We would only have to replace $r(t)$ by $r(t) - \lambda(t)$ everywhere.

an institution imposes some (non-competitive) intergenerational transfers. They will be studied in section 4.

In short, letting :

$$\xi(x,t) = c(x,t) - y_l(x,t)$$

we can draw the following table :

Table 1: Properties of different categories of economies.

	ξ is competitive	ξ is not necessarily competitive.
ξ is conservative	Pure exchange market economy *Samuelson (1958)* *Gale (1973)*	Pure exchange economy with intergenerational transfers
ξ is not necessarily conservative but $\text{Pop}(\xi,t) = \frac{1}{P(t)}(rK - \frac{dK}{dt})$	Closed market productive economy *Diamond (1965)* *Willis (1988)* *This article, section 3*	Closed productive economy with intergenerational transfers *Diamond (1965)* *Samuelson (1975)* *Willis (1988)*
ξ is not necessarily conservative	Open productive economy *Diamond (1965)*	Open productive economy with intergenerational transfers *This article, section 4*

Remark : Obviously the lower and the more to the right, the more general is the economy. Thus, Section 4 of this article deals with the most general economies, and in fact includes the closed market economies of section 3. However, since most articles have focused on the study of closed market economies, we decided to dedicate a distinct section to them.

2.2 Wealth held through a system of reallocation.

Our theoretical accounting framework is based on a concept of wealth held through a system of reallocation which we define as follows : the wealth held by an individual of age x at time t is equal to the present value of his net expectation of receipts from the system g. It is :

$$w_g(x,t) = \int_x^\omega \frac{p(u,t+u-x)}{p(x,t)} g(u,t+u-x) e^{-\int_t^{t+u-x} r(a)da} du \tag{1}$$

We define the aggregate wealth per capita at time t by the population weighted average :

$$W(g,t) = \frac{1}{P(t)} \int_0^\omega B(t-x) p(x,t) w_g(x,t) dx \tag{2}$$

It is important to note that this notion of wealth, which follows Lee's definition (1994a and b), is an expectation, therefore forward looking. It contrasts with the notion of assets used by Gale (1973) and Willis (1988) which looks backward. To look forward is essential in order to consider a non-trivial mortality pattern. Indeed, stochastic deaths are very easily included in a notion of expectation, but when one considers the notion of assets it raises some fundamental questions. When a person dies, his or her wealth (defined as an expectation) obviously becomes equal to zero since he or she is not going to be affected by the system of reallocation anymore. On the other hand, a person may die with some non-zero assets, and we may wonder what becomes of his or her assets (in the past literature this problem was avoided by considering a trivial mortality pattern: since people knew their age at death, they were supposed to behave rationally and die with zero assets). A solution might be to base the accounting on the cohort instead of the individual. This would work fine in the case of a competitive system of reallocation since $\mathrm{PV}(g,t) = 0$ implies that the aggregate assets held at death by the members of a given cohort are zero. This property no longer holds for more general systems, which makes the use of this notion of assets problematic in the present context.

Now, we can announce the main result of this theoretical part. The notion of aggregate wealth per capita held through a system of reallocation, as we defined it, is linked to the "conservativeness" and the "competitiveness" of this system as follows :

Theorem 2.2 *The wealth held through the system of reallocation g satisfies the equation :*

$$\frac{dW}{dt}(g,t) = (r(t) - n(t))W(g,t) + b(t)PV(g,t) - Pop(g,t)$$

where $n(t) = \frac{P'(t)}{P(t)}$ is the rate of population growth and $b(t) = \frac{B(t)}{P(t)}$ is the number of births per capita.

Comment :

This result is quite natural. It says that the wealth per capita increases because it earns returns at rate $r(t)$, decreases as a result of dilution due to the population growth, increases (or decreases) because some people are born with a positive (or negative) wealth[4], and decreases (or increases) as a consequence of the aggregate net flow of wealth. The mathematical proof is written in appendix 6.1.

2.3 An example: the steady states.

In steady states the dependence in t disappears and the definitions of PV and Pop may be simply written :

$$\mathrm{PV}(g) = \int_0^{\omega} e^{-rx} p(x)g(x)dx \tag{3}$$

$$\mathrm{Pop}(g) = b\int_0^{\omega} e^{-nx} p(x)g(x)dx \tag{4}$$

[4] A typical situation where people are born with non zero wealth would be the case where people have to participate in intergenerational transfers such as a Pay as You Go pension system. The rate of return for such systems is n (if productivity growth is zero) and therefore they may have a present value discounted at rate r, PV , not equal to zero.

where b is crude birth rate, that is the flow of births per capita, and n is the population growth rate. We may note that from the steady state expression (4) it is clear that the rate of population growth is always a solution for the internal rate of return earned on any conservative reallocation and therefore on any intergenerational transfer system. Theorem 2.2 implies that in a steady-state the wealth held through a system of allocation g must satisfy the equation :

$$(r - n)W(g) + b\text{PV}(g) - \text{Pop}(g) = 0$$

which shows how simply are related the concept of wealth, of "conservativeness" and of "competitiveness" in this particular case.

Consideration of steady-states also illustrates how we gain in generality when working with complex demographic hypotheses, rather than with simple ones as in the two age-group model. Indeed, from (3) and (4), we see that, with our continuous model, the vectorial spaces of competitive and conservative reallocation systems in steady-states are of infinite dimension as well as their intersection. This contrasts with the usual N age-group models where the two former spaces are of dimension $N - 1$ while their intersection is of dimension $N - 2$, if we are not in a Golden-Rule steady-state. In particular we see that in the case of two age-group models ($N = 2$), which includes almost all studies, the only system of reallocation which is competitive and conservative at the same time in a non Golden-Rule steady-state is the trivial one ($g = 0$). We could also have deduced this result from the following property conjectured by Lee (1993) and proved in appendix 6.2 :

Proposition 2.3 *In a non-Golden-Rule steady state any non-trivial system of reallocation which is both competitive and conservative must change sign at least twice during the life-cycle.*

(It is then obvious that in the context of a two age group models a system of reallocation cannot change sign more than once, since it takes only two values.)

This absence of non-trivial system of allocation which is competitive and conservative at the same time in a two age-group model explains why Samuelson (1958) had to introduce a 3 age-group model to permit the existence of a non Golden-Rule steady-state with some exchange between generations. Indeed, in a pure exchange competitive economy (that is, with no durable goods) reallocation between generations must be conservative since there is no possible investment (this is what Samuelson calls "market clearance") and must also be competitive (Samuelson's budget constraint). Thus, for Samuelson, the only way to leave the "biological rate of interest" and the no-exchange equilibrium was to consider at least a three age-group model. However, the three age-group model of Samuelson remained very restrictive since, in his model, for any values of r and n (with $r \neq n$) all intergenerational exchange that satisfies both the market clearance and the budget constraint must have the same shape. This limitation disappears in the continuous model where the variety of systems of reallocations which are competitive and conservative at the same time is infinite.

3 A closed market economy with capital.

From now on we will suppose that the population, but not necessarily the economy, is in a steady state, that is that $p(x,t)$, $n(t)$ and $b(t)$ do not depend on t. We suppose also that we are in a productive world where there is no technical progress and where production is an homogeneous function of Capital and Labor. Because the population is stable Labor is proportional to population size, and the production function can be written as :

$$F(t) = P(t)f(k(t))$$

where $k(t) = \frac{K(t)}{P(t)}$ is the capital per capita at time t. The function f is assumed to satisfy the usual conditions, $f \geq 0$, $f^{'} > 0$ and $f^{''} < 0$ plus the Inada conditions :

$$\lim_{k \to 0} f^{'}(k) = +\infty \quad \text{and} \quad \lim_{k \to +\infty} f^{'}(k) = 0$$

We will say that an economy is a closed market economy if the three following conditions are fulfilled :

- All that is produced is consumed or invested.
- Labor is paid its marginal product.
- Individual savings are competitive (agents having perfect foresight).

Recall that in our notation $c(x,t)$ and $y_l(x,t)$ designate the consumption and labor income of individuals of age x at time t, and $\xi(x,t)$ is their difference :

$$\xi(x,t) = c(x,t) - y_l(x,t)$$

The notation $W(c,t)$, $W(y_l,t)$ and $W(\xi,t)$ corresponds to the aggregate value per capita of wealth as defined by equation (2). $W(\xi,t)$ is the difference between the present values of expected consumption and expected income, which corresponds to the usual notion of wealth in the absence of bequests and other non-market transfers, as required by the third condition above.

With this notation the first condition may be written as :

$$f(k(t)) = \frac{dk}{dt}(t) + nk(t) + \text{Pop}(c,t) \tag{5}$$

which is the basic dynamic equation of Solow's growth model.

As we do not want to make additional assumptions about how productivity varies with age, we can use the second condition only at the aggregate level. It gives :

$$\mathrm{Pop}(y_l, t) = f(k(t)) - r(t)k(t) \tag{6}$$

with $r(t) = f'(k(t))$. [5]

The third condition says that at any time we have :

$$\mathrm{PV}(y_l, t) = \mathrm{PV}(c, t) \tag{7}$$

or, in other terms, that $\xi(x, t)$ is competitive in the sense we define in Section 2.

At this point these assumptions may seem restrictive since they rule out non-competitive intergenerational transfers or government taxes and transfers, but these will be examined in the last section of this article. On the other hand, the assumptions are more general than those in numerous articles such as in Galor and Ryder (1989) and Galor (1992). Indeed, following Samuelson (1958), Diamond (1965), Gale (1972, 1973), Balasko, Cass and Shell (1980), Tirole (1985), Weil(1987) and Lee (1994a and b), among others, we do not assume that the aggregate wealth held by individuals equals the amount of capital. Instead, we implicitly accept that there exists money or some institution that may support a government deficit. In the following we will call "balance" the difference between the aggregate wealth per capita and the capital per capita, which is $W(\xi, t) - k(t)$. This concept of balance will play a crucial role throughout our

[5] We can check easily that adding equations (5) and (6) we obtain :

$$\mathrm{Pop}(\xi, t) = (r - n)k + \frac{dk}{dt} = \frac{1}{P(t)}(rK - \frac{dK}{dt})$$

and that this kind of economy is indeed to be found in the second line of table 1.

analysis. Our aim is not to discuss how a non-zero balance can be introduced in the real world during what we may call a "pre-economic" period. But we will pay particular attention to the evolution of this balance during the economic era, when the three assumptions listed above are assumed to be satisfied.

In what follows, we will describe the static and dynamic properties of such economies. Before continuing, however, we claim the following result which is a simple consequence of our theoretical framework, and which will be useful in subsequent analysis.

Proposition 3.1 *In a closed market economy we have :*

$$\frac{d}{dt}(W(\xi,t)-k(t)) = (r(t)-n)(W(\xi,t)-k(t)) \tag{8}$$

Proof : Combining equations (5) and (6) we find :

$$\mathrm{Pop}(\xi,t) = \mathrm{Pop}(c,t) - \mathrm{Pop}(y_l,t) = -\frac{dk}{dt}(t) + (r(t)-n)k(t) \tag{9}$$

Also as a consequence of theorem 2.2 we have :

$$\mathrm{Pop}(\xi,t) = -\frac{dW}{dt}(\xi,t) + (r(t)-n(t))W(t) + b(t)\mathrm{PV}(\xi,t)$$

Since $\mathrm{PV}(\xi,t)=0$ by hypothesis (equation (7)), the subtraction of these two equations give the desired result.

3.1 The steady-states

In a steady state equation (8) may be simply written :

$$(r-n)(W(\xi)-k)=0$$

Therefore, using Gale's classification :

Proposition 3.2 *A steady state of a closed market economy is always either "balanced" $(W(\xi)=k)$ or "Golden-rule" $(r=n)$.*

This result has been obtained without strong assumptions about the behavior of agents. Indeed we only needed to assume that equation (7) is satisfied, which means that agents only borrow and lend at a competitive rate. However, although we have been able to describe some properties that must be satisfied by a steady-state, we do not yet know whether any such steady states exist. A priori, existence must depend on the agents' behavior, but we can obtain some results for a very general class of behavior, including rational behavior.

Suppose indeed that :

- $h1$) Agents choose their life cycle consumption in order to maximize a utility function U under the constraint $\mathrm{PV}(\xi,t)\leq 0$.
- $h2$) The utility and the production functions are such that in the hypothetical limit $k\to 0$ (and $r\to+\infty$), the aggregate wealth implied by $h1$ would exceed the value of capital.

Mathematically speaking the assumption $h1$ means that for all t the longitudinal consumption plan $c(x,t+x)$ is a solution of the program :

$$\operatorname*{Max}_{\mathrm{PV}(c,t)\leq\mathrm{PV}(y_l,t)} U(c(x))$$

Assumption $h2$ links together the properties of the individual's preferences and the properties of the production function. As k goes to zero the rate of interest goes to $+\infty$ and therefore we expect people to postpone their consumption, so that the wealth would be greater than the capital. However, at the same time that k tends to zero labor income decreases. It may happen, in some particular cases, that preferences for present consumption increase as income decreases in such a way as to offset the first effect. Assumption $h2$ is made in most articles on productive two age-group models (as in Diamond (1965), Tirole (1985), Weil (1987), etc.). However, Galor and Ryder (1989) showed examples where this assumption does not hold. They were able to establish some necessary conditions for $h2$ to hold, but they could not obtain an explicit sufficient condition. A sufficient condition may nevertheless be written as follows :

Proposition 3.3 *If preferences are additive and homothetic and if for any $\epsilon > 0$ there exists k_0 such that :*

$$\frac{f(k) - kf'(k)}{k} > e^{-\epsilon f'(k)}$$

for $k < k_0$, then $h2$ is always satisfied.[6]

The proof is in appendix 6.3. For our purposes, we only need to assume $h2$ for the following proposition which extends Gale's result (1973) :

Proposition 3.4 *If assumptions h1 and h2 are fulfilled, there always exist both a balanced steady-state and a Golden Rule steady state which are consistent both with our assumptions defining a closed market economy and with rational behavior of the agents.*

Moreover, if in the Golden-Rule $W(\xi) > k$ (resp: $W(\xi) < k$), then there exists a balanced equilibrium with $r < n$ (resp: $r > n$)[7].

[6] It is of interest to remark that the necessary condition obtained by Galor and Ryder (1989) for the two age group model becomes too strong when considering the continuous model.

[7] If assumption $h2$ does not hold then there may not exist a (non trivial) balanced steady-state when $W(\xi) < k$ in the Golden-Rule (see Galor and Ryder (1989)).

The proof is in appendix 6.4. Gale (1973) used the term "Samuelson" for the case where $W(\xi) > k$ in the Golden Rule steady-state, and "Classical" for the case where $W(\xi) < k$ in the Golden Rule steady-state. As Samuelson (1958) suggested, Golden Rule steady states of the former kind may be supported by the existence of money (with positive value). A Golden-Rule equilibrium of the latter kind would require some other institutional support allowing the society to keep a surplus of capital, since money of negative value is hardly imaginable.

There are several results concerning the welfare of agents. The following theorem, proved in appendix 6.5, extends the first result of Starret (1972) :

Theorem 3.5 *A Golden Rule steady state is Pareto optimal.*

It may happen coincidently that the Golden-Rule steady state is also a balanced steady state. This case corresponds to the "Goldenest Golden Rule" of Samuelson (1975). Indeed, more generally, we claim that :

Theorem 3.6 *If we denote by $U(n)$ the lifetime utility of individuals in the Golden-Rule steady state with rate of population growth n, and if $U(n)$ is continuously differentiable, then the first derivative of U always has the sign of the balance $W(\xi) - k$ of this Golden-Rule steady state.*

In other terms we have :

$$(W(\xi_{gr}) - k_{gr})\frac{d\,U}{d\,n} \geq 0$$

where the subscripts gr are introduced to remind that we refer here to the Golden-Rule steady state. Anticipating the result of corollary 4.4 this may also be written as :

$$(\mathrm{Pop}(c)(A_c - A_{y_l}) - k)\frac{d\,U}{d\,n} \geq 0$$

where A_c and A_{y_l} are the average ages of consumption and labor income, or also :

$$(A_{\tau^+} - A_{\tau^-})\frac{d\,U}{d\,n} \geq 0$$

where τ^+ and τ^- are the in-coming and out-going (competitive) intergenerational transfers. In particular if the Golden Rule is balanced then $\frac{dU}{dn} = 0$, which is the first order derivative condition that should be satisfied for an optimal population growth[8]. More precisely we may say that if the society is in a Classical (resp: Samuelson) Golden Rule steady-state the welfare of individuals could be improved, in the long term, by a slower (resp: faster) population growth. This result has been proved by Arthur and McNicoll (1978) for the case of an additive and atemporal utility function and is proved in a more general context in appendix 6.6. Willis (1988) was the first to connect the average result of Arthur and McNicoll (1978) to the balance measure, $W - k$. Willis and Kim (1988) derived a sufficient condition for an optimum in the three age-group case.

Coming back to the situation where the rate of population growth is exogenously fixed at a value n, we have the following properties which generalize Starret's second result (1972) :

Theorem 3.7 *A balanced steady state is Pareto efficient if $r \geq n$ and inefficient if $r < n$.*

We know from theorem 3.5 that the optimal steady-state is Golden-Rule. With this result, proved in appendix 6.7, we see that if the economy is initially in a balanced steady-state with $r < n$ a transition from the balanced steady state to the optimal one is feasible without diminishing the welfare of any individual (such a transition would, however, be classified as non-economic since it would necessarily violate one of our three hypothesis). On the other hand, if we are in a steady state with $r > n$ such a transition would necessarily be costly for some individuals.

[8]Samuelson (1975) intuitively interpreted this necessary condition, $\frac{dU}{dn} = 0$, as a sufficient condition for the existence of an optimal population growth rate. But Deardoff (1976), with Samuelson' acknowledgment (1976), showed that this rate of growth may also correspond to a welfare minimum.

These static properties do not imply that a market economy will converge to a Pareto optimal equilibrium, or even to Pareto efficient one. Samuelson's numerical example (1958), in a pure exchange economy with three age-groups, gives a situation where from almost every initial condition the economy will converge to an inefficient balanced equilibrium. Such a result shows the interest of studying the dynamic properties of overlapping generations market economies in general, as we aim to do in the following section.

3.2 Dynamics

Our framework allows us to gain some insight into the dynamics of such closed market economies, retaining the assumption of steady state population. Of course, the results that we present here are incomplete, with some questions remaining unsolved. We should not forget, however, that results on this topic were formerly available only for the two age-group model, with few exceptions.

Let us begin with the result of proposition 3.1 which says that in market economies we have :

$$\frac{d}{dt}(W(\xi,t)-k(t)) = (r(t)-n)(W(\xi,t)-k(t))$$

This can be solved as :

$$W(\xi,t)-k(t) = (W(\xi,0)-k(0))\exp\left(\int_0^t (r(a)-n)da\right) \tag{10}$$

Therefore we see that the evolution of the balance $W(\xi,t)-k(t)$ depends exclusively on the nature of the generalized integral :

$$\int_0^{+\infty}(r(t)-n)dt$$

The nature of this integral depends on the evolution of the rate of interest and thus indirectly on the agents' behavior. We may, however, obtain some results for the general case.

Proposition 3.8

1) If a program is balanced at some time then it remains balanced for ever.

2) The sign of the balance $W(\xi,t)-k(t)$ of a program is constant.

3) A balanced equilibrium with $r>n$ is not stable.

Comments :

The first point of this proposition is a generalization of the impossibility theorem of Samuelson (1958). It is also in Gale (1973) for a N age group model in a pure exchange economy. It says that a market economy cannot support the transition from a balanced state to an unbalanced one.

The second point is just more general. Together with the third point they give a partial answer to Gale's conjecture[9]. Indeed, if we are in what Gale calls the classical case (that is with $W(\xi)<k$ in the Golden-Rule steady-state) we know that there exists at least one balanced steady state with $r>n$. Our proposition says that in this case the economy will tend to move away (at least locally) from this balanced steady state. Moreover, if the initial conditions are such that $W(\xi,0)\geq k(0)$ then the balance will remain non positive (from point 2 of the proposition) and the economy will not converge towards the Golden-Rule steady states which would be characterized by a negative balance.

[9]Gale (1973) conjectured, for an economy with no durable good, that in the Classical case the balanced steady-state is unstable, the economy converging toward the Golden-Rule only if the initial balance is negative. He conjectured also that in the Samuelson case the economy always converges toward the balanced steady-state.

Proof of proposition 3.8 Points 1 and 2 come directly from equation (10). In point 3 by "not stable" we mean that for at least some initial conditions infinitely close to the steady-state the economic path will deviate from the steady-state. Now suppose that there exists a balanced steady state with a rate of interest $r^* > n$. Let us choose some initial conditions where the balance is close, but not equal, to zero, and where the rate of interest is close to r^* (so that $r - n > 0$). From equation (10) we know that an evolution where r remains close to r^* and the balance tends to zero is impossible. Thus we see that in this case, the economy which starts from initial conditions arbitrarily close to the balanced steady-state will move away from this steady-state.

4 General economies.

Although the study of market economies has been the main preoccupation for economists, in fact all real world economies include many kinds of transfers, not necessarily competitive, and are therefore non-market in our terminology. Examples of non-competitive reallocations of resources include child rearing, Pay-As-You-Go pension systems, familial intergenerational transfers, government taxes and transfers, etc. Indeed, such non-market transfers comprise by far the most important source of, or institutional support for, unbalanced economies (see Lee, 1994a).

The aim of this section is to show how our previous analysis can be easily extended to these general economies. In particular, we will describe the possible steady states and show how their characteristics are linked to the properties of "competitiveness" and "conservativeness" of the non-competitive systems of reallocation that may exist in these economies. We will also give a simple expansion which makes it possible to compute the wealth held through a system of reallocation from the moment of the age distribution of this system of reallocation. From this expansion we obtain the balance of a Golden-Rule economy, linking our results to those of Arthur and McNicoll (1978), Willis(1988) and Lee (1994a and b). We can also find the correction terms to be added when $r \neq n$.

Let us call τ the sum of all the non-market systems of reallocation, familial and governmental. We may think of τ as being determined and operationalized by some abstract (or real) unproductive institution, which gives $\tau^{+}(x,t)$ to any individual of age x at time t and collects $\tau^{-}(x,t)$, the net transfers being $\tau(x,t) = \tau^{+}(x,t) - \tau^{-}(x,t)$. In a closed economy, it is the nature of transfers that all that is given by some individuals is received by others, so that $\tau(x,t)$ must be conservative, satisfying $\text{Pop}(\tau, t) = 0$. However, we may think of different situations, such as an open economy, where $\tau(x,t)$ may include some government taxes collected for payment to some foreign economies (this would be the case for a country which has to pay interest on its external debt as in Diamond (1965)). In this case τ does not need to be conservative and to avoid any loss of generality we will not make any assumption on the value of $\text{Pop}(\tau, t)$ in the following.

Let us define here :

$$\theta(x,t) = \xi(x,t) - \tau(x,t) = c(x,t) - y_l(x,t) - \tau(x,t)$$

In absence of capital depreciation we have:

$$\frac{dk}{dt}(k) + nk(t) = rk(t) - \text{Pop}(\theta, t)$$

The hypothesis that individuals behave competitively (a part from the fact that they participate in the system of reallocation τ) means that $\text{PV}(\theta, t) = 0$. Thus using the theorem 2.2 we get :

$$\frac{d}{dt}(W(\theta,t) - k(t)) = (r(t) - n)(W(\theta,t) - k(t)) \tag{11}$$

In a steady-state all the dependence in t disappears and this equation simply becomes $(r - n)(W(\theta) - k) = 0$. Therefore

Proposition 4.1 *A steady-state must be either :*

- *Golden-Rule*

- *Non-Golden Rule (and "non consensual", in the sense defined below) with a balance given by :*

$$W(\xi) - k = W(\tau) = \frac{1}{n-r}\Big(bPV(\tau) - Pop(\tau)\Big) \tag{12}$$

Outside of Golden-Rule, the aggregate individual wealth must equal the sum of the capital and the aggregate institutional-transfer wealth. In the Golden-Rule this equality is not necessary. The difference may be supported for example by the presence of money, or more generally by some other competitive intergenerational transfers. Proposition 4.1 is obviously the generalization of Proposition 3.2. Most of the previous results can be extended in the same way. Indeed we see at a glance that equations (8) and (11) are identical. In the case of a general economy with institutional intergenerational transfers, the important variable is no longer the balance $W(\xi, t) - k$, but $W(\theta) - k$, which is also $W(\xi) + W(\tau) - k$, the difference between the aggregate wealth and the sum of the capital per capita and the (institutional) transfer wealth. To avoid any confusion we will call this difference the "consensual-wealth" to insinuate that this kind of wealth, which adds to the wealth held in form of assets and to the institutional transfers wealth, has to arise from a social consensus. The "consensual wealth" includes the wealth held in money or bonds and also the wealth that might be held from other intergenerational transfers than τ. We will say that a program is "non-consensual" if the "consensual wealth" equals zero. Theorems 3.5, 3.7 and Proposition 3.8 become then :

Theorem 4.2 *In a general economy with fixed (and unchangeable) institutional transfers and taxes we know that : (i) A Golden-Rule steady-state is Pareto-optimal.*

(ii) A "non consensual" steady state is Pareto Efficient if $r \geq n$ and inefficient if $r < n$.

(iii) If a program is "non consensual" at some time then it remains "non consensual" for ever.

(iv) The sign of the "consensual wealth" is constant.

(v) A "non consensual" steady-state with $r > n$ is not stable.

Proof The proof of (i) is exactly the same as the proof of Theorem 3.5. For the proof of (ii) we can literally follow the proof of theorem 3.7 replacing $c(x,t)$ and $c_0(x)$ by $c(x,t) - \tau(x)$ and $c_0(x) - \tau(x)$, respectively. Substituting ξ for θ in the proof of proposition 3.8 proves (iii), (iv) and (v).

We have not yet given any method to compute the balance of an economy in the Golden-Rule. Theoretically this may be done with an argument of continuity. At this point we must be careful since we know that there may exist some discontinuity when r equals n due to the theoretical viability of money or of competitive intergenerational transfers in the Golden-Rule which is impossible in other steady-states. Here we resort to the artifact of considering the Golden-Rule steady-state as a non-monetary economy with institutionally fixed intergenerational transfers. Let τ_{gr} denote the sum of all the different kinds of intergenerational transfers in the Golden-Rule steady-state (including transfers of money) [10]. The Golden-Rule steady-state is the limit when $r \to n$ of non-monetary economies with fixed intergenerational transfers τ_{gr}. We know then that the balance in the Golden-Rule must equals the limit of the balances of these non-monetary economies and is therefore given by the limit of (12) when $r \to n$ and when τ is taken equal to τ_{gr}.

[10] Rigorously $\tau_{gr}(x)$ is equal to $c_{gr}(x) + i_{gr}(x) + m_{gr}(x) - y_{gr}(x)$ where $c_{gr}(x)$, $i_{gr}(x)$, $m_{gr}(x)$, and $y_{gr}(x)$ are respectively the consumption, investment, net credit transaction, and total income at age x, in the Golden Rule steady-state.

In general we would like to be able to compute the balance from the parameters which are easily observable, such as the rate of interest and the age-distribution of the system of reallocation. Although (12) and the remark above give us a theoretical solution, this is not necessary the most efficient and informative way to compute this balance. In particular, when r and n are close and, even worse, when we have to consider the limit $r \to n$ to obtain the balance in the Golden-Rule, the apparent singularity of expression (12) when $r = n$ would very probably introduce some lack of precision when used empirically. It would be particularly subject to criticism since we know from Lee (1994a and 1994b) that the balance in the Golden-Rule has a very simple expression. Thus the aim of the end of this section is to develop an exact expansion which expresses the balance, or more generally the wealth held through a system of reallocation, as a clearly continuous function of $r - n$, measuring in some sense the distance to the Golden-Rule, and the moments of the age distribution of the system of reallocation we consider. The first step consists in the following result :

Lemma 4.3 *For any system of reallocation $g(x)$ (constant over time) such that $Pop(g) \neq 0$ we have :*

$$bPV(g) = e^{(n-r)A_g} Pop(g)\Big(1 + \frac{(n-r)^2}{2}\sigma_g^2 + \cdots + \frac{(n-r)^k}{k!}\mu_g(k) + \cdots\Big) \qquad (13)$$

Where A_g, σ_g^2, $\mu_g(3)$, $\mu_g(4)$, etc. are the (population weighted) mean, variance and higher moments of the age distribution of $g(x)$. They are defined by :

$$A_g = \frac{\int_0^\omega x e^{-nx} p(x) g(x) dx}{\int_0^\omega e^{-nx} p(x) g(x) dx}$$

$$\sigma_g^2 = \frac{\int_0^\omega (x - A_g)^2 e^{-nx} p(x) g(x) dx}{\int_0^\omega e^{-nx} p(x) g(x) dx}$$

$$\mu_g(k) = \frac{\int_0^\omega (x - A_g)^k e^{-nx} p(x) g(x) dx}{\int_0^\omega e^{-nx} p(x) g(x) dx}$$

Proof : For any δ we compute :

$$\begin{aligned} \mathrm{PV}(g) &= e^{(n-r)\delta} \int_0^{+\infty} p(x) e^{(n-r)(x-\delta)} e^{-nx} g(x) dx \\ &= e^{(n-r)\delta} \sum_{k=0}^{+\infty} \frac{(n-r)^k}{k!} \int_0^{+\infty} p(x)(x-\delta)^k e^{-nx} g(x) dx \end{aligned} \tag{14}$$

Choosing $\delta = A_g$ as defined above gives formula (13).

Usually we are interested in systems of reallocation which may be positive at some ages and negative at others. For this kind of system lemma (4.3) is not very satisfactory because the definitions of the mean, variance, and other moments given above do not then have a clear physical meaning. Also, for any conservative system, $\mathrm{Pop}(g)$ will be zero and the lemma will not apply. But note that any system of reallocation g can be decomposed into $g = g^+ - g^-$ where $g^+(x)$ describes the resources flowing to the agent (such as social security benefits) while $g^-(x)$ describes resources flowing from the agent (such as social security taxes paid). Then the previous result can be applied, yielding :

Corollary 4.4 *In a steady-state, for any system of reallocation g we have :*

$$\begin{aligned} W(g) = \ & \tfrac{1}{n-r}\left(Pop(g^+)(e^{(n-r)A_{g^+}} - 1) - Pop(g^-)(e^{(n-r)A_{g^-}} - 1)\right) \\ & + \tfrac{(n-r)}{2}\left(Pop(g^+)\sigma^2_{g^+} e^{(n-r)A_{g^+}} - Pop(g^-)\sigma^2_{g^-} e^{(n-r)A_{g^-}}\right) \\ & + \cdots \\ & + \tfrac{(n-r)^{k-1}}{k!}\left(Pop(g^+)\mu_{g^+}(k) e^{(n-r)A_{g^+}} - Pop(g^-)\mu_{g^-}(k) e^{(n-r)A_{g^-}}\right) \\ & + \cdots \end{aligned} \tag{15}$$

This expression gives an empirically efficient method for computing the wealth held through a system of reallocation. We may apply it to compute various relevant economic values. For instance, according to (12), setting $g(x) = \tau(x)$ we obtain the balance of the economy. Another application would be to choose $g^{+}(x) = c(x)$ and $g^{-}(x) = y_l(x)$ which leads to computation of the wealth $W(\xi)$ from the age-specific profiles of consumption and labor income. Indirectly we obtain a method to estimate the value of capital per capita.

If we develop the first term of (15) our expression takes the form of a perturbation expansion in $r - n$, which gives some insights on the validity of the Golden-Rule approximation and on how fast it deteriorates. Concerning the particular case of the Golden-Rule steady-state we can easily check from (15) that, for any system of reallocation, $W(g) \rightarrow \text{Pop}(g^{+})A_{g^{+}} - \text{Pop}(g^{-})A_{g^{-}}$ when $(r - n) \rightarrow 0$. So, for example, we find that in the Golden Rule $W(c) = \text{Pop}(c)A_c$ and $W(y_l) = \text{Pop}(y_l)A_{y_l}$ and that the balance is given by $W(\xi) - k = \text{Pop}(\tau^{+})A_{\tau^{+}} - \text{Pop}(\tau^{-})A_{\tau_{-}}$ results which are consistent with the results found previously by Lee (1994a and 1994b).

5 Conclusion

The extensive literature on overlapping generation models is rich and productive, yet it suffers from its reliance on simplistic demographic assumptions which are largely unnecessary. The past literature has mainly assumed only two age groups and perfect survival until the end of the second of these. Theoretical results for two age groups sometimes do not generalize, and without mortality one cannot study the implications of its change. Such a crude model cannot even simultaneously accommodate dependent childhood, productive mid years, and retirement. Any kind of empirical implementation of these models is virtually impossible.

This paper adopted a continuous age distribution with an arbitrary age schedule of mortality. The core economic model was standard. The new concept of reallocation system is sufficiently broad to include the real world variety of individual and institutional mechanisms, ranging from real capital formation through credit transactions to non-market transfers through the family or the government. A simple two-by-two categorization of these according to whether they are competitive and/or conservative captures their analytically critical features. A notation based on this categorization facilitates compact expressions.

Life cycle age profiles of planned survival-weighted earning and consumption, in conjunction with the age distribution of the population, give rise to an aggregate demand for wealth. If this demand is exactly satisfied by aggregate holdings of capital, the economy is said to be balanced. More generally, however, the demand for wealth differs from the size of the capital stock, and wealth is additionally held in some other form–typically as transfer wealth. The difference between wealth and capital per head is called the balance of the economy.

We have derived a considerable number of theoretical results in this paper. Some of these extended the findings of other studies to a more general demographic context or to economies with capital; some provide simpler proofs; and some established new results. We began with a very general expression for the evolution of wealth in non-steady state economies and non-stable populations. We then considered closed market economies. The previous result, applied to a market economy with a stable population, led to a simple expression for the evolution of the balance of the economy. This in turn implied that all steady state market economies must be either golden rule or balanced. We also showed that with rational agents with additive homothetic preferences and some conditions on the production function, there will always exist one balanced equilibrium and one golden rule equilibrium. Next we established some welfare results for closed market economies. A golden rule steady state was shown to be Pareto Optimal. When the balance is positive, which occurs when the average age of consuming exceeds the average age of earning, then a steady state with a higher rate of population growth will permit a higher level of welfare in a comparative static sense, and when the balance is negative, then a steady state with more rapid population growth would be beneficial. A balanced steady state is Pareto Efficient if r exceeds n, but inefficient if r is less than n. Finally we derive some dynamic results for closed market economies, showing that a balanced economy cannot become unbalanced, and that in an unbalanced economy, the sign of the balance cannot change. A balanced economy with r greater than n is not stable, despite being Pareto Efficient. This set of results achieves a synthesis, generalization and extension of a substantial literature in this area.

In these hypothetical closed market economies, it is not entirely clear how it is possible for unbalanced economies to occur and be sustained by market institutions. In the real world, non-market interage transfers are pervasive, and lead to positive or negative transfer wealth, and thereby readily create and support unbalanced economies. Unfunded public sector pension programs, or familial support by adults of their elderly parents, for example, create enormous positive transfer wealth which sustains positive balance in the economy. Intended or unintended bequests, and publicly funded education, create very substantial negative transfer wealth, tending to create a negative balance (see Lee, 1994a and b). We considered economies with transfers of this general sort, and showed how earlier results generalized. A steady state economy with transfers must either be golden rule or have a balance related in a specific way to the age profile of transfers. Results on the optimality, efficiency and stability of the different kinds of steady-state were extended straightforwardly, as well as the results on the dynamic. Further analysis led to a perturbation expansion about the golden rule case which provided a convenient basis for evaluating the balance in the golden rule case as well as outside of it. In golden rule, the balance is given by the average per capita inflow of transfers times the average age of receiving a transfer, minus the corresponding product for the outflow of transfers, a result which holds for open economies as well as closed ones. In closed economies, the inflows and outflows of transfers must be equal, so the sign of the balance of the economy is positive when the population-weighted average age of receiving transfers exceeds that of making them, as appears to be the case in the U.S. and some other industrial nations (Lee, 1994a and b).

Economists should not be put off by the apparent complexity of realistic demographic models, models which in principle should permit a much greater degree of generality and relevance to real world phenomena and policy problems. The same model that has been empirically implemented elsewhere to study the consequences of population aging has here been related to a deeper theoretical literature. We have shown that such models remain tractable, and that comparative static, dynamic and welfare theoretic results can be obtained.

6 Appendix

6.1 Proof of theorem 2.2

From equation (1) we may compute :

$$(\frac{d}{dx} + \frac{d}{dt})w_g(x,t) = (-g(x,t) + r(t)w_g(x,t) - \frac{p_x^{'} + p_t^{'}}{p}(x,t)w_g(x,t)) \tag{16}$$

Multiplying both sides by $B(t-x)p(x,t)$ and integrating between 0 and ω we get :

$$\int_0^{\omega} B(t-x)(\frac{d}{dt} + \frac{d}{dx})(p(x,t)w_g(x,t))dx = -P(t)\mathrm{Pop}(g,t) + r(t)P(t)W(g,t) \tag{17}$$

On the other hand let us note :

$$f(t) = P(t)W(g,t) = \int_0^{\omega} B(t-x)p(x,t)w_g(x,t)dx$$

We have :

$$f'(t) = \int_0^\omega \Big(B'(t-x)p(x,t)w_g(x,t)dx + B(t-x)\frac{d}{dt}(p(x,t)w_g(x,t)) \Big) dx$$

Integrating by parts the first term of the integral yields :

$$f'(t) = B(t)p(0,t)w_g(0,t) + \int_0^\omega B(t-x)(\frac{d}{dt} + \frac{d}{dx})(p(x,t)w_g(x,t))dx$$

The first term on the right hand side of this equation is precisely $B(t)\mathrm{PV}(g,t)$. Then, using equation (17) we obtain :

$$f'(t) = r(t)P(t)W(g,t) - P(t)\mathrm{Pop}(g,t) + B(t)\mathrm{PV}(g,t)$$

and since $W(g,t) = \frac{f(t)}{P(t)}$ implies that $\frac{dW(g,t)}{dt} = -nW(g,t) + \frac{1}{P(t)}f'(t)$ we get exactly the announced result.

6.2 Proof of proposition 2.3

We show that there is no function $g(x)$ which solves :

$$\int_0^\omega e^{-rx}p(x)g(x)dx = \int_0^\omega e^{-nx}p(x)g(x)dx = 0 \qquad (18)$$

which does not change sign more than once and which is not identically zero. Let us imagine that there exists such a function and, for example, that $r > n$ and that $g(x) \leq 0$ for $x \leq x_0$ and $g(x) \geq 0$ for $x \geq x_0$. We would have :

$$\int_0^{x_0} e^{-rx}p(x)g(x)dx \leq e^{(n-r)x_0} \int_0^{x_0} e^{-nx}p(x)g(x)dx$$

$$\int_{x_0}^{\omega} e^{-rx} p(x) g(x) dx \leq e^{(n-r)x_0} \int_{x_0}^{\omega} e^{-nx} p(x) g(x) dx$$

If $g(x)$ is not identically zero one of these inequalities must be strict. Adding these two inequalities would lead to a contradiction of equation (18). Thus we see that the existence of such a function is impossible. The cases $r < n$ or $g(x) \geq 0$ for $x \leq x_0$ and $g(x) \leq 0$ for $x \geq x_0$ may be treated similarly.

6.3 Proof of proposition 3.3

As remarked by Kessler and Masson (1988) any additive homothetic intertemporal utility function may be written as :

$$U(c) = \int_0^{\omega} p(x) \alpha(x) u(c(x)) dx$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. A simple calculation shows that the consumption pattern that follows from utility maximization satisfies $c(x) = c(0) e^{(r-\lambda_x)x/\gamma}$ where $\lambda_x = -\frac{1}{p(x)\alpha(x)} \frac{d}{dx}(p(x)\alpha(x))$. The value of $c(0)$ is determined by the constraint $\text{PV}(c) = \text{PV}(y_l)$ and we have :

$$\frac{1}{b}\text{Pop}(c) = \frac{\int_0^{\omega} e^{-nx} p(x) e^{(r-\lambda_x)x/\gamma} dx}{\int_0^{\omega} e^{-rx} p(x) e^{(r-\lambda_x)x/\gamma} dx} \text{PV}(y_l)$$

It follows that for any age a such that $p(a) \neq 0$ there exists r_0 such that $\frac{1}{b}\text{Pop}(c) \geq e^{(r-n)a}\text{PV}(y_l)$ for $r \geq r_0$. In particular, since the income profile (given by the labor productivity) is supposed to be fixed, we know that there exists $\epsilon > 0$ such that for r large enough $\text{Pop}(c) \geq e^{\epsilon r}\text{Pop}(y_l)$.

Assumption $h2$ says that $W(\xi)$ should be greater than k in the hypothetical limit $k \to 0$. We know from theorem 2.2 that $(r-n)W(\xi) = \text{Pop}(\xi)$ so assumption $h2$ is also equivalent to: "In the hypothetical limit $r \to +\infty$ we should have $\text{Pop}(c) - \text{Pop}(y_l) > (r-n)k$."

We have seen that for some positive ϵ we have in the limit $r \to +\infty$:

$$\text{Pop}(c) - \text{Pop}(y_l) > (e^{\epsilon r} - 1)\text{Pop}(y_l) \geq e^{\epsilon_1 r}\text{Pop}(y_l) = e^{\epsilon_1 f'(k)}(f(k) - kf'(k)) \qquad (19)$$

Thus, assuming that preferences are additive and homothetic, the condition which says that for any $\epsilon > 0$ we must have :

$$\frac{f(k) - kf'(k)}{k} > e^{-\epsilon f'(k)} \qquad (20)$$

for k small enough is enough to ensure assumption $h2$.[11]

6.4 Proof of proposition 3.4

Assuming that agents behave rationally, for any hypothetical steady state, characterized by a value of capital per capita k and a rate of interest $r = f'(k)$, there corresponds an income profile $y_l(x)$ determined by the marginal productivity of Labor and a consumption profile $c(x)$ which maximize $U(c(x))$ under the constraint $\text{PV}(c) = \text{PV}(y_l)$. We will note $\xi(x) = c(x)) - y_l(x))$ where $-\xi(x)$ is in some sense the "rational" investment of individuals of age x from their labor income. An hypothetical steady state will actually be a feasible steady state if the desired investment, $-\text{Pop}(\xi)$, is equal to the investment needed to maintain the capital per capita at his level. This means that a steady-state is feasible if and only if :

$$-\text{Pop}(\xi) = (n - r)k$$

Let us call $z(r) = -\text{Pop}(\xi) - (n - r)k$ which is the difference between the rational investment and the necessary investment to support a steady state. We will show that $z(r)$ has always

[11] In fact we could have shown that (19) is true for any ϵ_1 smaller than the difference between the maximum age at death and the minimum age of non-zero productivity. We would obtain a weaker sufficient condition supposing only that (20) has to be true for some ϵ smaller than this age gap.

at least two roots when r varies in $[0, +\infty]$, and therefore that there are always two possible steady-states.

The root $r = n$ corresponding to the Golden-Rule steady state is obvious. Indeed when $r = n$ the application PV and Pop are proportional and :

$$\text{Pop}(\xi) = \text{Pop}(c) - \text{Pop}(y_l) = b\text{PV}(c) - b\text{PV}(y_l) = 0$$

since $\text{PV}(c) = \text{PV}(y_l)$ is the constraint of the individual maximization program.

It remains to see that there is at least one other root. The consumption being non negative we have $-\text{Pop}(\xi) \leq \text{Pop}(y_l) = f(k) - rk$. Thus we see that $z(r) \leq f(k) - nk$ and since $f'(k) \to 0$ when $k \to +\infty$ we have :

$$\lim_{r \to 0} z(r) = -\infty \tag{21}$$

From theorem 2.2 we know that in a steady state we have $\text{Pop}(\xi) = (r-n)W(\xi)$ (since $\text{PV}(\xi) = 0$ from the individual budget constraint). Therefore :

$$z(r) = (n - r)(W(\xi) - k) \tag{22}$$

and hypothesis $h2$ implies that $z(r) \leq 0$ when $r \to +\infty$.

The function $z(r)$ is continuous, non positive when $r \to +\infty$ and when $r \to 0$ and is equal to zero when $r = n$. Moreover we know from equation (22) that the derivative of z in $r = n$ is given by :

$$\frac{dz}{dr}|_{r=n} = -(W(\xi) - k)|_{r=n}$$

which is the opposite of the balance in the Golden Rule equilibrium. Thus in the classical case where $W(\xi) < k$ in the Golden rule we know that there exists at least one root of $z(r)$ greater than n and in the "Samuelson's case", when $W(\xi) > k$ in the Golden rule we know that there exists one root smaller than n.

6.5 Proof of theorem 3.5

By claiming that a Golden-Rule steady-state is Pareto optimal we mean that agents in such a steady-state have a higher lifetime utility that in any other steady state. Indeed let us call k_g, $c_g(x)$, y_{l_g} the capital per capita, labor income and consumption profiles of the Golden-Rule steady state and k and $c(x)$, $y_l(x)$ the values in another steady-state. To show that $U(c(x)) \leq U(c_g(x))$ we only need to prove that $\mathrm{Pop}(c) \leq \mathrm{Pop}(y_{l_g})$ since $c_g(x)$ is by assumption a solution of the program :

$$\underset{\mathrm{Pop}(c_g) \leq \mathrm{Pop}(y_{l_g})}{\mathrm{Max}} U(c_g(x))$$

(in the Golden Rule the function Pop and PV are proportional). But it is well known that the Golden Rule steady state is the steady state that maximizes the aggregate consumption. So we have $\mathrm{Pop}(c) \leq \mathrm{Pop}(c_g) \leq \mathrm{Pop}(y_{l_g})$, which completes the proof.

6.6 Proof of theorem 3.6

Let us note $c_n(x)$ and y_{l_n} the consumption an labor income of individuals of age x in the Golden-Rule of rate of population growth (and rate of interest) n. We know that c_n maximizes the utility function under the constraint $\mathrm{PV}(c_n) = \mathrm{PV}(y_{l_n})$. Thus, any consumption pattern such that $\mathrm{PV}(c) < \mathrm{PV}(c_n)$ would make individuals worse off and any better consumption pattern will have to satisfy $\mathrm{PV}(c) > \mathrm{PV}(c_n)$. Consequently, as we assumed $U(n)$ to be continuously differentiable, the sign of $\frac{dU}{dn}(n)$ will be the same as the sign of $\mathrm{PV}(\frac{\partial c}{\partial n})$.

We know that for every Golden Rule steady state we have $\mathrm{PV}(c_n) = \mathrm{PV}(y_{l_n})$. Differentiating both sides of this equality and using the fact that for a positive system of reallocation $\frac{d}{dn}(\mathrm{PV}(g)) = -A_g \mathrm{PV}(g) + \mathrm{PV}(\frac{\partial g}{\partial n})$ we obtain :

$$- A_c \mathrm{PV}(c) + \mathrm{PV}(\frac{\partial c}{\partial n}) = -A_{y_l} \mathrm{PV}(i) + \mathrm{PV}(\frac{\partial y_l}{\partial n}) \tag{23}$$

$b\mathrm{PV}(\frac{\partial y_l}{\partial n})$ or also $\mathrm{Pop}(\frac{\partial y_l}{\partial n})$ is the variation of aggregate income at constant structure by age and is given by :

$$b\mathrm{PV}(\frac{\partial y_l}{\partial n}) = \mathrm{Pop}(\frac{\partial y_l}{\partial n}) = \frac{d}{dn}(f(k) - nk) = (f^{'}(k) - n)\frac{\partial k}{\partial n} - k = -k \tag{24}$$

which corresponds to the classic effect of capital dilution of the growth model of Solow. Using the result of Lee (1994) or anticipating the result of corollary 4.4 we know that :

$$W(\xi) = (A_c - A_{y_l})\mathrm{Pop}(c) \tag{25}$$

From (23) (24) (25) we get $\mathrm{PV}(\frac{\partial c}{\partial n}) = \frac{1}{b}(W(\xi) - k)$ which completes the proof.

6.7 Proof of theorem 3.7

Let us prove first that a balanced equilibrium program with $r > n$ is Pareto-efficient, or in other words that there does not exist any transition that makes nobody worse off and at least one individual better off. For this we show that the existence of such a transition would lead to some inconsistency.

Let us call k_0, r_0, c_0 the values of the capital per capita, rate of interest and consumption of the steady state, before the transition, and call PV_0 the function PV obtained for $r = r_0$ Applying the result of theorem (2.2) we know that :

$$\frac{dW_0}{dt}(c,t) = (r_0 - n)W_0(c,t) + b\mathrm{PV}_0(c,t) - \mathrm{Pop}(c,t)$$

where we call W_0 the notion of wealth defined as in equation (2) for $r = r_0$. With equation (5) we get :

$$\frac{d(W_0(c,t)-k)}{dt} = (W_0(c,t)-k)(r_0-n) + r_0 k - f(k) + b\mathrm{PV}_0(c,t)$$

Calling :

$$\alpha(t) = (W_0(c,t)-k) - (W_0(c_0)-k_0)$$

we have :

$$\frac{d}{dt}(\alpha e^{(n-r_0)t}) = e^{(n-r_0)t}(r_0 k - f(k) + b\mathrm{PV}_0(c,t) - (n-r_0)(W_0(c_0)-k_0)) \qquad (26)$$

or also given the fact that $\alpha(0) = 0$, per definition, and that $(n-r_0)(W_0(c_0)-k_0) = r_0 k_0 - f(k_0) + b\mathrm{PV}_0(c_0)$ (since the zero indices correspond to a steady state program) we obtain :

$$\alpha(t) = \int_0^t e^{(r_0-n)(t-u)}(r_0 k(u) - f(k(u)) - (r_0 - f(k_0)) + b\mathrm{PV}_0(c,u) - b\mathrm{PV}(c_0))du \qquad (27)$$

Now note that a simple study of the variation of the function $r_0 k - f(k)$ shows that for all k we have $r_0 k - f(k) \geq r_0 k_0 - f(k_0)$ since $r_0 = f'(k_0)$ and $f''(k) \leq 0$. Remark also that a transition that makes one person better off and nobody worse off must pass by a point where $\mathrm{PV}_0(c,t) > \mathrm{PV}_0(c_0)$ and be such that $\mathrm{PV}_0(c,t) \geq \mathrm{PV}_0(c_0)$ for every t. Thus, since $r_0 > n$, we see from equation (27) that such a transition transition would lead to :

$$\lim_{t\to+\infty}(W_0(c,t)-k)(t) = +\infty$$

which is physically impossible (it would mean that the expected consumption of agents, discounted at the rate r_0, would tend to infinity).

To prove that a balanced steady-state with $r < n$ is not Pareto efficient we must construct a transition that makes nobody worse off and some persons better off. The idea is very simple. Imagine that starting from this balanced steady-state we add after some time $t = t_0$ an infinitesimal conservative intergenerational transfer going from the younger to the elderly. The capital per capita is not affected by such a transition since we make only a reallocation of resources between generations. It is easy to check that after the transition is achieved people will have a higher utility than those who were alive before the transition since from an individual viewpoint this intergenerational transfer is like an infinitesimal investment at a rate $n > r$. For people alive during the transition the situation is even better since they receive all the benefit of this kind of "investment" without having paid all the contributions.

References

[1] W. B. Arthur and G. McNicoll. Samuelson, population and intergenerational transfers. *International Economic Review*, 19:241–246, 1978.

[2] Y. Balasko, D. Cass, and K. Shell. Existence of competitive equilibrium in a general overlapping-generations model. *Journal of Economic Theory*, 23:307–322, 1980.

[3] Y. Balasko and K. Shell. The overlapping-generations model, i: The case of pure exchange without money. *Journal of Economic Theory*, 23:281–306, 1980.

[4] Y. Balasko and K. Shell. The overlapping-generations model, ii: The case of pure exchange with money. *Journal of Economic Theory*, 24:112–142, 1981.

[5] Y. Balasko and K. Shell. The overlapping-generations model. iii. the case of log-linear utility functions. *Journal of Economic Theory*, 24:143–152, 1981.

[6] A. V. Deardoff. The growth rate for population: comment. *International Economic Review*, 17:510–515, 1976.

[7] P. A. Diamond. National debt in a neoclassical growth model. *American Economic Review*, 55:1126–1150, 1965.

[8] J. Esteban, T. Mitra, and D. Ray. Efficient monetary equilibrium: An overlapping generations model with nonstationnary monetary policies. *Journal of Economic Theory*, 64, 1994.

[9] D. Gale. *Mathematical Topics in Economic Theory and Computation*, chapter On Equilibrium Growth of Dynamic Economic Models, pages 84–98. Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1972.

[10] D. Gale. Pure exchange equilibrium of dynamic economic models. *Journal of Economic Theory*, 6:12–36, 1973.

[11] O. Galor. A two-sector overlapping-generations model: a global characterization of the dynamical system. *Econometrica*, 60(6):1351–1386, November 1992.

[12] O. Galor and H. E. Ryder. Existence, uniqueness, and stability of equilibrium in an overlapping-generations model with productive capital. *Journal of Economic Theory*, 49:360–375, 1989.

[13] O. Galor and H. E. Ryder. Dynamic efficiency of steady-state equilibria in an overlapping-generations model with productive capital. *Economic Letters*, 35:385–390, 1991.

[14] D. Kessler and A. Masson. *Modeling the accumulation and distribution of wealth.*, chapter Wealth Distributional Consequences of Life Cycle Models, pages 287–318. Clarendon press, Oxford, 1988.

[15] R. D. Lee. Age structure, intergenerational transfers and economic growth: An overview. *Revue Economique*, 31(6):1129–1156, 1980.

[16] R. D. Lee. Research proposal to the National Institute of Aging, September 1993.

[17] R. D. Lee. Population age structure, intergenerational transfers, and wealth: A new approach, with applications to the u.s. *Journal of Human Resources*, 1994a.

[18] R. D. Lee. *Demography of Aging*, chapter The Formal Demography of Population Aging, Transfers, and the Economic Life Cycle, pages 8–49. National Academy Press, 1994b.

[19] P. A. Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66(6):467–482, December 1958.

[20] P. A. Samuelson. The optimum growth rate of a population. *International Economic Review*, 16(3):531–538, 1975.

[21] P. A. Samuelson. The optimum growth rate of a population: agreement and evaluations. *International Economic Review*, 17:516–525, 1976.

[22] D. A. Starret. On golden rules, the "biological theory of interest", and competitive inefficiency. *Journal of Political Economy*, 80:276–291, 1972.

[23] J. Tirole. Assets bubbles and overlapping generations. *Econometrica*, 53:1071–1100, 1985.

[24] P. Weil. Confidence and the real value of money in an overlapping generations economy. *Quarterly Journal of Economics*, 102:1–22, 1987.

[25] R. J. Willis. *Economics of Changing Age Distributions in Developed Countries*, chapter Life Cycles, Institutions, and Population Growth: A theory of the Equilibrium Interest Rate in an Overlapping-Generations Model, pages 106–138. Oxford University Press, 1988.