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Cohort Quantum as a Function of Time-dependent Period Quantum for Non-repeatable Events

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In a recent article in this journal expressions were derived for the translation of period quantum indicators for age-specific non-repeatable events (e.g. first marriage, births by parity) into cohort quantum indicators.[1] These expressions extend Ryder's well-known translation formulae which apply to age-specific repeatable events (e.g. births irrespective of parity) only.

One of the conclusions of the recent article is that when the period age pattern of the event (period tempo) does not change over time and period quantum deceases linearly with time, the quantum for cohort g is overestimated by the period quantum as measured in period g; however, this conclusion was found to hold for low values of the period quantum only. For high values, the period quantum in year g *underestimates* the quantum for the cohort born in year g, in spite of a fall in the period quantum. For instance, Figure 1 in the original article shows that when period quantum is 0.75 (i.e. 75 per cent of the synthetic cohort ever experience the event, a value which is based on the set of age-specific rates for the period in question) and period quantum falls by two percentage points annually, cohort quantum would be 0.89.[2] No explanation could be given in the original paper for this counter-intuitive behaviour of the cohort quantum.

In the present paper we will show that the peculiar behaviour of the cohort quantum is caused by the fact that the quadratic approximation from which it has been calculated does not hold: the Taylor series expansion of the general formula does not converge. We present simulation results which indicate that this non-convergence, and consequently the breakdown of the quadratic approximation formula, will occur in many cases of interest. Finally, we derive translation formulae for time-dependent cohort and period quantum which do not suffer from the type of non-convergence referred to above.

I. TRANSLATION FORMULAE FOR NON-REPEATABLE EVENTS

Let $m(t, x)$ be a time- and age-specific occurrence-exposure rate for some non-repeatable event, with x and t representing age and time, respectively. We use the term 'quantum' to denote the proportion of individuals in a real or synthetic cohort who ever experience the event, and the term 'tempo' for the timing of this event during the life course. Assuming piece-wise constant intensities, the period quantum equals

$$Q_p(t) = 1 - \exp[-A(t)], \tag{1}$$

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[1] N. Keilman, 'Translation formulae for non-repeatable events', *Population Studies*, **48** (1994), pp. 341–358.

[2] Values for the first two moments of the schedule of age-specific rates were assumed equal to 30 years and 1000 years^2, respectively.

where $\sum_{x=0}^{w-1} m(t,x) = A(t)$ is the period sum of rates for a particular year t, and w is the maximum age. Similarly, we define the cohort quantum as

$$Q_c(g) = 1 - \exp[-B(g)], \tag{2}$$

where $\Sigma_{x=0}^{w-1} m(g+x, x) = B(g)$ represents the cohort sum of rates for a certain cohort $g = t - x$. We introduce the age-specific proportions $a(t,x)$ of the sum of period rates $A(t)$ by $a(t,x) = m(t,x)/A(t)$, and the kth moment about zero of the age schedule $a(t,x)$ as $M_k(t) = \Sigma x^k a(t,x)$. These moments describe the tempo of the event. Next, Taylor series approximations for $A(g+x)$ and $a(g+x,x)$ about $t = g$ enable us to write the cohort sum of rates $B(g)$ as a function of the period sum of rates for the year $t = g$, the moments of the age schedule, and their derivatives with respect to time:

$$B(g) = A(g)\{1 + M_1'(g) + \tfrac{1}{2}M_2''(g) + \ldots\} + A'(g)\{M_1(g) + M_2'(g) + \tfrac{1}{2}M_3''(g) + \ldots\}$$
$$+ A''(g)\{\tfrac{1}{2}M_2(g) + \tfrac{1}{2}M_3'(g) + \tfrac{1}{4}M_4''(g) + \ldots\} + \ldots \tag{3}$$

For repeatable events, this is one of Ryder's basic translation formulae, in which $A(t)$ and $B(g)$ serve as indices for period and cohort quantum respectively (i.e. in the case of age-specific fertility, period Total Fertility and Completed Cohort Fertility).[3] For non-repeatable events, the quantum is a function of the rate sum, and we use Equations (1) to (2) to find

$$Q_c(g) = 1 - \{1 - Q_p(g)\}^{E_1(g)} \exp\{-E_2(g)\}, \tag{4}$$

where we write in short $E_1(g) = 1 + M_1'(g) + \tfrac{1}{2}M_2''(g) + \ldots$, and

$$E_2(g) = \frac{Q_p'(g)}{1 - Q_p(g)}\{M_1(g) + M_2'(g) + \tfrac{1}{2}M_3''(g) + \ldots\}$$
$$+ \left[\frac{Q_p''(g)}{1 - Q_p(g)} + \left\{\frac{Q_p'(g)}{1 - Q_p(g)}\right\}^2\right]\{\tfrac{1}{2}M_2(g) + \tfrac{1}{2}M_3'(g) + \tfrac{1}{4}M_4''(g) + \ldots\} + \ldots$$

Equation (4) (numbered (13) in the original article) is the general formula for cohort quantum as a function of period quantum and period tempo in the case of non-repeatable events.

In the present article, the focus is on time-dependent cohort and period quantum, assuming that period tempo is constant. (Expressions for cohort quantum when period quantum is constant and period tempo changes with time, and for cohort tempo, are discussed in the original article.) In that case all derivatives with respect to time of all moments M vanish and Equation (4) reduces to

$$Q_c(g) = 1 - \{1 - Q_p(g)\} \exp\left[-\sum_{i=1}^{\infty} \frac{M_i}{i!} A^{(i)}(g)\right]. \tag{5}$$

In the original article, the case was investigated in which period quantum changes linearly with time (and period tempo is constant). This implies that $Q_p'(g) = Q_p'$ is constant, and all other derivatives of $Q_p(g)$ are zero, so that (5) becomes

$$Q_c(g) = 1 - \{1 - Q_p(g)\} \exp\left[-\sum_{i=1}^{\infty} \frac{M_i}{i}\left(\frac{Q_p'}{1 - Q_p(g)}\right)^i\right]. \tag{6}$$

[3] N. B. Ryder, 'The process of demographic translation', *Demography*, **1** (1964), pp. 74–82; 'Components of temporal variations in American fertility', in R. W. Hiorns (ed.), *Demographic Patterns in Developed Societies* (London: Taylor & Francis, 1980), pp. 15–54.

In addition, in the original article it was assumed that Q'_p is so small that terms of third and higher orders can be ignored. In that case it was found that

$$Q_c(g) = 1-\{1-Q_p(g)\}\exp\left[-M_1\frac{Q'_p}{1-Q_p(g)}-\tfrac{1}{2}M_2\left\{\frac{Q'_p}{1-Q_p(g)}\right\}^2\right]. \tag{7}$$

Equation (7) shows the counter-intuitive behaviour referred to above. For rising period quantum ($Q'_p > 0$) the period quantum in year g underestimates the cohort quantum of cohort g, as expected. Similarly, a falling period quantum leads to a situation in which the period quantum overestimates the corresponding cohort quantum – but only when period quantum values are below $1+\frac{1}{2}Q'_p M_2/M_1$, as was argued in Section III.2 of the original article. When period quantum exceeds the latter value, the cohort quantum is *underestimated* by the period quantum: the denominator of the quadratic term in the exponent in Equation (7) becomes so small, that this term itself exceeds, in absolute value, the first term. For negative values of Q'_p, the first term in the exponent is positive. Thus, for low values of $Q_p(g)$, the exponent is positive too, but when $Q_p(g)$ exceeds the critical level of $1+\frac{1}{2}Q'_p M_2/M_1$, the exponent becomes negative, and period quantum will underestimate cohort quantum. This situation will arise particularly when M_2 is large compared to M_1. The counter-intuitive behaviour of the cohort quantum for negative slopes of the period quantum is clearly illustrated by Figure 1, in Keilman's original paper.

II. SIMULATIONS

In order to investigate the behaviour of Equation (7) further, we carried out a number of simulations based on age-specific first marriage rates for males aged 15–59 in Norway for the period 1971–75 (see Appendix of the original article for the data). Each simulation was performed for one particular choice for the values of $Q_p(g)$ and Q'_p. The cohort quantum was computed in three different ways.

(1) According to Equation (7).

(2) According to Equation (6). Compared to formula (7), Equation (6) contains all powers of $\{Q'_p/(1-Q_p(g))\}$, instead of just the first two. We took as many terms in the summation as were necessary for convergence (increment smaller than 10^{-4}). Moments were computed on the basis of the base rates.

(3) By direct observation: for each of the 60 years of the simulation period, the base rates were appropriately scaled up or down, in accordance with the values of $Q_p(g)$ and Q'_p, and the assumptions of constant period tempo and linear behaviour of period quantum. Next, the cohort quantum was computed on the basis of the rates located on the diagonal.

A selection of the results is displayed in Table 1. For an increase of one per cent in period quantum, there is a reasonable agreement between the results of Equation (7) and the 'true' values of the cohort quantum, i.e. those of the direct computations. For period quantum values up to 0.7, Equation (7) underestimates the true value by between one and four percentage points. When the period quantum is 0.8 or larger, the true value is overestimated by up to five percentage points when Equation (7) is used. Note that Formula (6) does not converge for period quantum values of 0.5 or higher. When period quantum falls by one per cent per year, Equation (7) overestimates the true cohort quantum value considerably; at period quantum levels above 0.7 the excess can lie between 10 and 30 percentage points. The situation for $Q'_p = -0.02$ is even worse and bears no resemblance to reality any longer (results not shown in Table 1): for example, when period quantum is 0.8, Equation (7) results in $Q_c(g) = 0.9561$, whereas the true

Table 1. *Cohort quantum values computed on the basis of three different approaches*

	$Q'_p = 0.01$			$Q'_p = -0.01$		
Q_p	Equation (7)	Equation (6)	Direct	Equation (7)	Equation (6)	Direct
0.1	0.3807	0.3923	0.3924	*	*	*
0.2	0.4786	0.4935	0.4935	*	*	*
0.3	0.5756	0.5955	0.5955	0.0349	0.0149	0.0136
0.4	0.6711	0.7002	0.6993	0.1423	0.1152	0.1150
0.5	0.7640	nc	0.8000	0.2545	nc	0.2155
0.6	0.8519	nc	0.8883	0.3762	nc	0.3159
0.7	0.9294	nc	0.9425	0.5197	nc	0.4165
0.8	0.9842	nc	0.9501	0.7190	nc	0.5172
0.9	0.9999	nc	0.9501	0.9780	nc	0.6184
0.95	1.0000	nc	0.9501	1.0000	nc	0.6692
0.99	1.0000	nc	0.9501	1.0000	nc	0.7100

* These combinations of a low initial period quantum and a linear decline lead to negative values for the cohort quantum. Clearly, the assumption of a linear fall of period quantum for a period of 60 years is not tenable at low values of initial period quantum.

nc, No convergence.

value is 0.2361. (When $Q'_p = +0.02$, Equation (7) but *under*estimates the true cohort quantum value by up to nine percentage points for period quantum values up to 0.5, but *over*estimates it by between two and five percentage points for period quantum values between 0.6 and 1.0.) Additional simulations showed that Equation (7) gives a reasonable estimate of the true cohort quantum (that is, the overestimation was two percentage points at most) in the case of a decreasing period quantum provided the slope was not steeper than between 0.2 ($Q_p = 0.9$) and 0.8 ($Q_p = 0.5$) percentage points per year.

The conclusion of these simulations is that Equation (7) results in reasonable values for the cohort quantum for a period quantum which increases by up to two percentage points annually. When the fall in period quantum is less than 0.8 percentage points annually, Equation (7) may give a reasonable estimate, provided that the initial period quantum is not too high. When period quantum decreases by one percentage point or more, Equation (7) cannot be used. However, Equation (6), based on the full series, cannot be used either: convergence is only reached for period quantum values of 0.5 or lower (assuming an increase or decrease in period quantum by one percentage point per year). Many variables of interest (first marriages, first births) would have quantum levels of between 0.5 and 1.0. Convergence is problematic because higher-order moments of the age schedule increase so quickly. We found the following values for the first six moments of the 1971–75 schedule of first marriage rates: 28.76, 878, 2.87E4, 1.01E6, 3.80E7, and 1.54E9. In spite of the fact that $\{Q'_p/(1-Q_p(g))\}^i$ becomes smaller with increasing i, multiplication by M_i/i results in a divergent series. Thus, if a third-degree term were to be included in Equation (7), this would not improve the situation.

III. CONSTANT PERIOD TEMPO AND A POLYNOMIAL CHANGE IN THE PERIOD RATE SUM

Instead of assuming a particular development in the period quantum, as was done in the original paper, one could also consider a certain form of the period rate sum. In this section we look at the situation in which the period rate sum is a low-degree polynomial of time. Period tempo is assumed constant, as before.

Suppose first that the rate sum $A(t)$ is a linear function of time with slope A'. In this case Equation (3) becomes simply $B(g) = A(g)+M_1 A' = A(g+M_1)$, and thus, from (1) and (2),

$$Q_c(g) = 1-\{1-Q_p(g)\}\exp[-M_1 A'] = Q_p(g+M_1). \tag{8}$$

In other words, assuming a linear change in the period sum of rates and constant period tempo, the cohort quantum for the cohort born in a certain year equals simply the period quantum M_1 years later, where M_1 represents the mean age of the period rate schedule (which, for non-repeatable events, does not correspond to the mean age at which the members of the synthetic cohort experience the event). The shift over an age interval of length M_1 occurs irrespective of how steeply the period rate sum increases or decreases (provided the period quantum remains between 0 and 100 per cent, of course). Ryder has obtained the same result for the case of repeatable events.[4] Here we have shown that it holds for non-repeatable events as well.

No summation is involved in the evaluation of Equation (8), and therefore convergence problems of the kind discussed in the previous section will not occur. Furthermore, when A' is negative, $Q_p(t)$ decreases exponentially, and therefore $Q_p(g+M_1) = Q_c(g) < Q_p(g)$. Hence the cohort quantum of cohort g will always be overestimated by the period quantum in year g, irrespective of the level of the latter. Thus Equation (8) does not show the kind of counter-intuitive behaviour that is given by Equation (7).

Is there a similar simple relationship between cohort and period quantum when the rate sum is a higher-degree polynomial of time? In case $A(t)$ is quadratic, Equation (4) becomes

$$Q_c(g) = 1-\{1-Q_p(g)\}\exp[-M_1 A'(g)-\tfrac{1}{2}M_2 A'']. \tag{9}$$

It is straightforward to prove that Equation (9) equals $Q_p(g+M_1)$, provided that $M_1 = \sqrt{M_2}$. Whether this relationship between first and second moment holds is an empirical question, but in the data set for first marriages among males in Norway between 1961 and 1990 we found that $\sqrt{M_2}$ was only up to three per cent higher than M_1.

In general, when the rate sum follows an nth degree polynomial function of time, and $M_i = (M_1)^i$ $(i = 2, 3, \ldots, n)$, then $Q_c(g) = Q_p(g+M_1)$.[5] However, because of the well-known instability of higher moments and higher-order polynomials we have not investigated this further. In particular, we do not know how much $Q_c(g)$ will deviate from $Q_p(g+M_1)$ when M_i is different from $(M_1)^i$.

IV. CONCLUSIONS

In this paper we have discussed some formulae which translate period quantum and period tempo of an age-specific non-repeatable event (for instance, first marriage or first birth) into cohort quantum. We have investigated the behaviour of the cohort quantum under the assumption of a linear change in the period quantum and a constant period tempo. We have shown that the peculiar behaviour of the cohort quantum which was

[4] Ryder (1964), *loc. cit.* in fn. 4, p. 78.

[5] When $A(t)$ is an nth degree polynomial in t, then

$$A(g+M_1) = A(g)+\sum_{i=1}^{n}\frac{(M_1)^i}{i!}A^{(i)}(g).$$

Comparison with Equation (5) shows that $(M_1)^i = M_i$ $(i = 1, 2, \ldots, n)$ is a sufficient condition for $B(g) = A(g+M_1)$ to hold. From the latter equality it follows that $Q_c(g) = Q_p(g+M_1)$.

found in an earlier paper on the subject, i.e. that the period quantum in a certain year underestimates the quantum for the cohort born in that year provided the period quantum decreases linearly, is explained by the unjustified neglect of higher-order terms in one of the expressions derived in the original article. We have presented simulation results which show that the non-convergence of the corresponding Taylor series expansion will occur in many cases of interest. As an alternative to the assumption of a linear change in the period quantum, we have proposed expressions for cohort quantum based on an assumption for the development over time of the period sum of rates. When the rate sum follows a straight line (and period tempo is constant), the cohort quantum for a birth cohort born in a given year equals the period quantum observed M_1 years later, where M_1 is the mean age of the period schedule of age-specific rates. We have also given results for the situation in which the rate sum follows a second-degree polynomial. In that case, too, the cohort quantum equals the period quantum M_1 years later, provided that the first moment of the schedule of age-specific rates equals the square root of the second moment, respectively. This result can be generalized to the case in which the period sum of rates follows a polynomial function of time of any order.