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Author(s): William Siler

Source: *Ecology*, Vol. 60, No. 4 (Aug., 1979), pp. 750-757

Published by: Ecological Society of America

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A COMPETING-RISK MODEL FOR ANIMAL MORTALITY¹

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Abstract. A three-component competing-risk model for animal mortality is presented, in which the additive hazards include a new model, dominant during the prematurity period; a constant hazard, dominant during the period of maturity; and the conventional Gompertz hazard, dominant during senescence. A good fit of the model is obtained to survival data for a variety of species, with both laboratory and field data being represented. Interpretation of the model parameters in terms of animal adjustment to hazards is offered.

Key words: *competing risk; hazard; life table; mortality; survival curve.*

INTRODUCTION

Since Pearl and Miner's (1935) unsuccessful attempt to find a mathematical model which would fit survival curves for a variety of species, a number of such models has been constructed, including those of Szilard (1959), Sacher and Trucco (1962), Johnson (1963), Noble and Hayes (1964), Curtis (1966), Burch (1967), and Forbes et al. (1970). Most of those models deal primarily with senescent mortality. The oldest such model (Gompertz 1825) retains a fair degree of acceptance; Calloway (1967) wrote that summation of various causes of senile decay composed of "random events constantly accelerated" tends cumulatively toward a Gompertz-like distribution. Reliability theory has been little used in the case of animal mortality; Gehan and Siddiqui (1973) discussed the fitting of some standard failure rates to animal survival data, and a work by Proschan and Serfling (1974) contained a number of theoretical papers concerned with the application of reliability theory to mortality. There presently exists a considerable volume of data against which models of animal mortality can be tested. Pearl and Miner (1935), Deevey (1947) and Laws (1968) presented life tables for a variety of species.

More than 40 yr ago, Pearl and Miner (1935) noted that there are three general types of animal survival curves: I (concave downward); II (log linear); and III (concave upward). Type I is the typical "wear-out" failure pattern; Type II is the common constant hazard or exponential survival pattern; and Type III can be considered the "early failure" pattern.

In attempting to find a satisfactory mathematical model for mortality, candidate distributions may be classified as corresponding to one or more of Pearl and Miner's (1935) three types. Most of the above models are of Type I or II hazards. Type III (decreasing) hazards have been discussed, although little has been presented in the way of specific mathematical models. Davis (1952) and Barlow and Proschan (1965) noted

that there is experimental evidence for hazard functions which decrease initially. Williams (1966) suggested that evolutionary selection should result in an orderly reduction in mortality with increasing age. Proschan (1963) noted that decreasing failure rates may result from summing the effects of concurrent differing constant hazards. Two workers, G. C. Williams and J. M. Emlen (*personal communication*) suggested a decreasing hazard model based on evolutionary principles; this model worked fairly well for data on primitive man but could not be fitted to crude fish-survival data, mainly due to the zero survivorship expected from the early failure period.

The Weibull distribution has attractive features which are well stated by David (1974) and by Pinder et al. (1978). These include the ability to provide a hazard function that increases, is constant, or decreases (depending on shape constant), thus corresponding to all three of Pearl and Miner's (1935) types. There is, however, a problem with the use of the Weibull distribution for early failure. Some organisms, notably fish, exhibit extremely high failure rates during the larval and juvenile periods with fairly stable long-term survivorship thereafter (Marr 1956). This pattern cannot be fitted with the Weibull distribution, again due to the zero survivorship expected from the early failure period.

In fact, empirically obtained hazard functions from reliability data are well known to decline initially, level off, and then increase. Empirically obtained animal survival curves yield hazard rates with the same characteristics (Emlen 1970). This paper will suggest that these characteristics occur as a result of three competing risks: a hazard that decreases as a result of animal adjustment to its environment, particularly as a result of maturation; a constant hazard, reflecting those hazards to which the animal does not adjust; and an increasing hazard occurring as a result of senescence. For the first of these, a new distribution is proposed, unique in that from this hazard alone a finite survivorship is expected at infinite time; for the second, the conventional constant hazard model is as-

¹ Manuscript received 13 October 1976; accepted 5 October 1978.

sumed; and for the third, the 150-yr-old Gompertz (1825) distribution is retained, partly because of its theoretical basis and its consistency with the derivation of the decreasing hazard function, and partly because of its satisfactory fit to actual animal survival data.

PROPOSED COMPETING-RISK MODEL

Suppose that an individual organism adapts to a specific hazard to life in such a way that the rate of change of the hazard equals an adjustment constant times the magnitude of the hazard itself. For a decreasing hazard (Type III), the adjustment constant is negative; for a constant hazard (Type II), the adjustment constant is zero; and for an increasing hazard (Type I), the adjustment constant is positive. We now assume that there are three (and only three) adjustment patterns occurring concurrently, one for each of the above cases; we further assume that these three risks are competing, but noninteracting. This approach then yields the total hazard function to be:

$$r(t) = a_1 e^{-b_1 t} + a_2 + a_3 e^{+b_3 t}. \quad (1)$$

Here r is the total hazard function, a_i is the hazard weight (or initial hazard), and b_i is the adjustment constant. By differentiating the total hazard rate with respect to time t , it can be seen that the hazard will eventually increase monotonically; by setting time $t = 0$, it can be seen that the hazard rate will initially decrease provided that the product $a_1 b_1 > a_3 b_3$.

Hazard for immature animals

Survivorship from the second and third of the individual risks discussed above are well known, being zero at infinite time. However, survivorship from the first of the above risks is qualitatively different from that expected from other proposed early failure distributions. If $\ell_1(t)$ is the survivorship at time t , then for the decreasing hazard function,

$$\ell_1(t) = \exp\{(-a_1/b_1)[1 - \exp(-b_1 t)]\}. \quad (2)$$

At infinite time, the survivorship would be:

$$\ell_1(\infty) = \exp(-a_1/b_1). \quad (3)$$

A nonzero survivorship is then expected at infinite time from the decreasing hazard. This survival pattern itself does not, of course, make much sense; however, because the first risk is only one of three competing risks, the second and third of which have expected survivorship zero at infinite time, the net survivorship at infinite time would be zero. The biological significance of the parameters a_1 and b_1 may be better seen by noting that a_1 represents the initial value of the hazard at the time zero, and that $1/b_1$ is the time constant with which the hazard is reduced. In particular, for the case where the first hazard represents that of immaturity, if

$$1/b_1 = T_m \quad (4)$$

then T_m is the time constant with which maturity is approached.

While the most common use of this decreasing hazard would be to account for the hazard due to immaturity, it can also be used (as mentioned above) in a more general sense for other hazards to which an animal adjusts successfully. An example is the hunting hazard to which male deer become subject after reaching legal hunting age. (This use of an additional hazard of the form of the "immaturity" hazard would require specifying the age at which the decreasing hazard becomes effective.)

An experimental check on the validity of the decreasing hazard model is shown in Fig. 8. This figure, taken from data reported in Fineman et al. (1974), shows the model fitted to laboratory survival data for the freshwater killifish *Oryzias latipes*, the Medaka.

It appears then that the desirable mathematical characteristics of the Weibull hazard function are vitiated in the early period of animal survival curves, when a decreasing failure rate is normally observed. The new decreasing hazard function here proposed, unusual in that from this risk alone a nonzero survivorship is expected, does have desirable features when considered as one of several competing risks.

Hazard for mature animals

Because time-varying hazards due to immaturity and senescence are separated by our model, the hazard for mature animals is simply a constant:

$$\mu_2 = a_2,$$

with survivorship

$$\ell_2 = \exp(-a_2 t). \quad (6)$$

Hazard for senescence

For simplicity, we here retain the conventional Gompertz model, except for a parameter transformation which makes the parameters easy to interpret. The Gompertz model

$$\mu_3 = a_3 \exp(b_3 t), \quad (7)$$

gives a survivorship

$$\ell_3 = \exp\{(a_3/b_3)[1 - \exp(b_3 t)]\}. \quad (8)$$

The associated probability density function is

$$f(t) = a_3 \exp(b_3 t) \exp\{(a_3/b_3)[1 - \exp(b_3 t)]\}, \quad (9)$$

and the log of the density function, L , is:

$$L = \ell_n a_3 + b_3 t + (a_3/b_3)[1 - \exp(b_3 t)]. \quad (10)$$

The modal survival time, T_s , if only μ_3 were operating, is then obtained by equating dL/dt to zero, yielding

$$T_s = (1/b_3) \ell_n(b_3/a_3). \quad (11)$$

An approximation to survival time variance σ^2 is obtained from $(-d^2L/dt^2)^{-1}$ with $t = T_s$, or

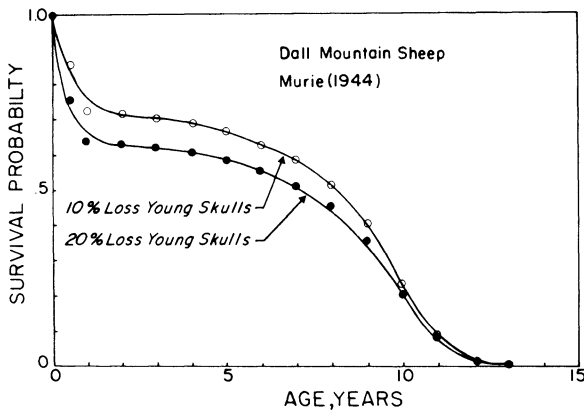


FIG. 1. Proposed mortality model fitted to survival curve data for the Dall Mountain Sheep (*Ovis dalli dalli*).

$$\hat{\sigma}^2 = (1/b_3^2). \quad (12)$$

(This approximation is valid for the normal distribution and is relatively insensitive to skewness but sensitive to kurtosis, as can be easily verified. Negative kurtosis will result in an overestimation of variance; positive kurtosis results in an underestimation of variance.) Replacing a_3 and b_3 by T_s and σ

$$\mu_3 \doteq (1/\hat{\sigma}^2)\exp[(t - T_s)/\sigma], \quad (13)$$

$$\ell_3 \doteq \exp\{\exp(T_s/\hat{\sigma}) - \exp[(t - T_s)/\hat{\sigma}]\}. \quad (14)$$

An approximate method for estimating the Gompertz parameters by desk calculator from real survival data follows from the above. Because in real survival data there is little difference between mean and modal survival times, a_3/b_3 being small,

$$T_s \doteq \bar{T}, \quad (15)$$

in which \bar{T} is the mean survival time if only μ_3 were operative. Given this approximation, then a simple approximation to parameter b_3 is given by

$$b_3 \doteq 1/\hat{\sigma}, \quad (16)$$

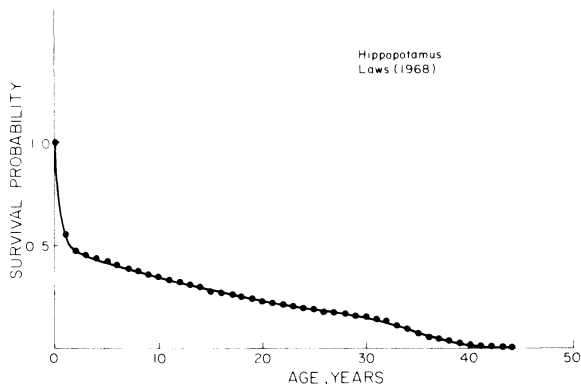


FIG. 2. Proposed mortality model fitted to survival curve data for the hippopotamus (*Hippopotamus amphibius*).

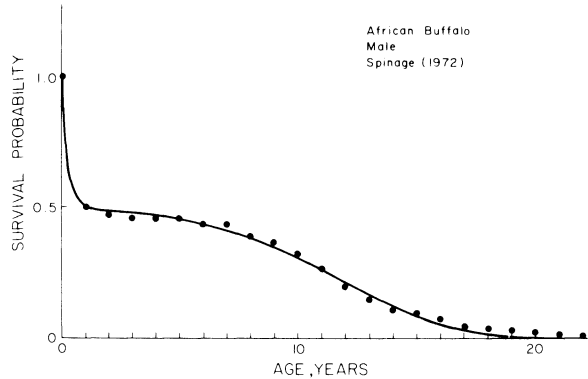


FIG. 3. Proposed mortality model fitted to survival curve data for the African Buffalo (*Syncerus caffer*).

in which T is the set of measured survival times. A simple approximation to a_3 follows:

$$a_3 \doteq (1/\hat{\sigma})\exp(-\bar{T}/\hat{\sigma}). \quad (17)$$

Combined mortality and survivorship

Recalling our postulate that the three types of maturity are additive, we have

$$\mu = \mu_1 + \mu_2 + \mu_3, \quad (18)$$

$$\ell = \ell_1 \cdot \ell_2 \cdot \ell_3, \quad (19)$$

with μ_i and ℓ_i given above.

These three components of mortality may be related to those noted by Pearl and Miner (1935): Type I, with a long plateau at survivorship nearly unity, followed by a relatively swift descent; Type II, log-linear; and Type III, with survivorship decreasing swiftly during the early time period, concave upwards, and then approaching zero roughly asymptotically. Deevey (1947) associated the Type III curve with heavy juvenile mortality. We note that Pearl and Miner's (1935) Type III curve corresponds to the distribution above for the hazard due to immaturity; their Type II curve corresponds to the hazard for the mature animal; and their



FIG. 4. Proposed mortality model fitted to survival curve data for the female zebra (*Equus burchelli boehmi*).

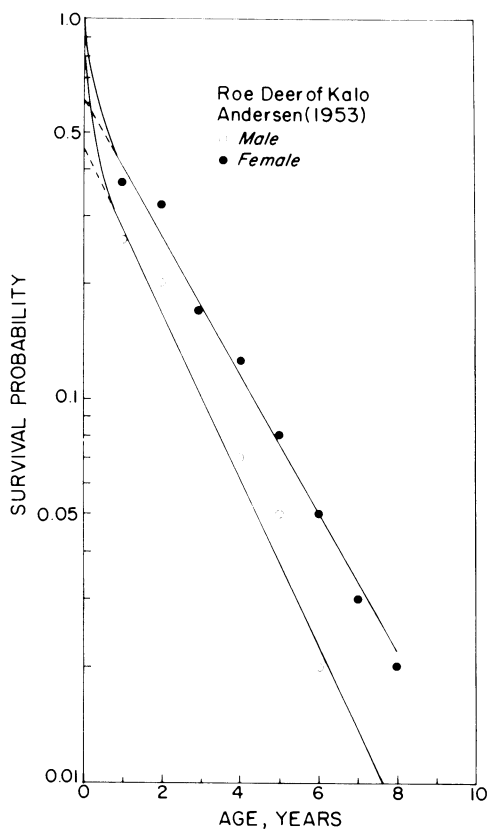


FIG. 5. Proposed mortality model fitted to survival curve data for the Roe Deer (*Capreolus capreolus*) of the Kalo.

Type I curve corresponds to the hazard due to senescence.

The combined mortality, obtained by simple addition of the three component hazards, then declines sharply during the infant and juvenile period, levels off during adulthood, and finally climbs sharply during senescence. These are characteristics of animal mortality pointed out by Emlen (1970).

DATA ANALYSIS

The model was tested by fitting it to a number of measured survival curves in the literature for a variety of species and by examining the "best fit" parameters to see if they made biological sense.

Model fitting was accomplished by a least-squares nonlinear regression program using a Gauss-Newton approach. Derivatives were determined numerically. The method is similar to that of Marquardt (1963). (For those few cases in which individual survival times are available, maximizing the log likelihood function would be preferable.)

In few cases were sufficient data available to evaluate the entire model. Frequently a large hazard to the mature animal (presumably predation or starvation) so reduced the sample at the older ages that information

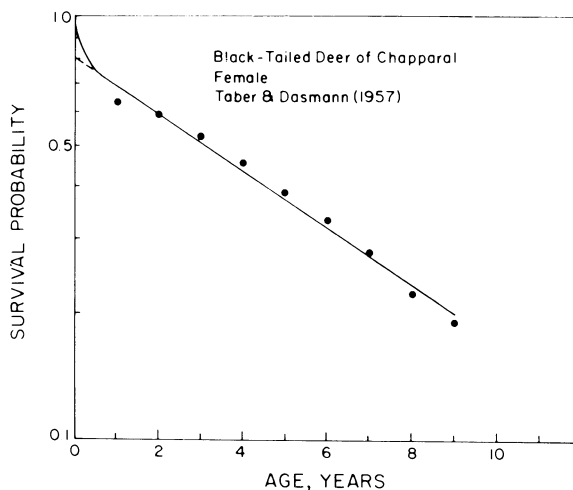


FIG. 6. Proposed mortality model fitted to survival curve data for the Black-tailed Deer (*Odocoileus hemionus columbianus*) of the chaparral.

on senescence could not be obtained. In most cases insufficient data were available during the prematurity period to evaluate both μ_0 and T_m , the two parameters for the immaturity hazard; in such cases the fraction surviving the immaturity hazard, $l_1(\infty)$, was determined.

Results of fitting the model to some representative life tables for vertebrates are given in Table 1, and in Figs. 1 through 8. Standard errors of the parameter values, obtained from linear model theory, are given in Table 2. These results will be separately discussed.

DISCUSSION

Dall Mountain Sheep, Ovis dalli dalli (Murie 1944)

Figure 1. The skulls of the very young animals in Murie's collection are fragile and, as Deevey (1947) remarked, are probably underestimated in the data. For the first run a 10% loss in the youngest age group (0–6 mo) was assumed. These data gave a 78% survival from the hazard of immaturity with a maturity time constant of 0.94 yr. The constant hazard of maturity is quite small (0.008), indicating a relative immunity of the mature, healthy animal to predation, though Murie (1944) stated the major cause of death is predation by wolves. Apparently the wolves prey mainly on the older, weaker animals. Modal survival time from the senescence hazard is ≈ 10 yr, with a standard deviation of ≈ 1.5 yr. A second run was made with an assumed 20% loss of the youngest skulls. Survivorship from immaturity was reduced from 78% to 64%; the maturity time constant was reduced from 0.94 to 0.47 yr, as mentioned above. In all probability the maturity time constant is not far removed from the time to sexual maturity; probably, then, Murie (1944)

TABLE 1. Model parameter values.

Animal	Error of fit σ	Initial hazard μ_0	Maturity time constant T_m	Immaturity survivors $\ell_1(\infty)$	Constant hazard k	Senescence modal span T_s	Senescence SD $\hat{\sigma}_s$
Dall Sheep							
10% correction	0.002†	0.26	0.94	0.78	0.008	10	1.5
20% correction	0.011†	0.94	0.47	0.64	0.011	10	1.5
Hippotamus	0.0046†	1.17	0.58	0.51	0.039	37.5	3.2
African Buffalo male	0.019†	2.72	0.26	0.50	0	12.6	4.0
Zebra, female	0.024†	0.51	0.28	0.87	0.076	16.8	1.8
Kalo Deer male	0.17‡	*	*	0.46	0.50	*	*
female	0.08‡	*	*	0.62	0.41	*	*
Chaparral Deer female	0.05‡	*	*	0.80	0.153	*	*
British Blackbird	0.17‡	*	*	0.88	0.59	*	*
Medaka							
XY	0.021†	0	*	1.0	0.0011	417	29
YY	0.020†	0.0115	24.5	0.75	0	353	73
YY (androgen)	0.021†	0.026	23.5	0.54	0	331	92

† RMS absolute error.

‡ RMS error relative to survival probability.

* Cannot be determined from the data. For the hazard of immaturity, model reparameterized in terms of immaturity survivorship $\ell_1(\infty)$ instead of μ_0 and T_m . If the life span was not long enough to determine the senescence hazard, that hazard was set to zero.

has not greatly underestimated the quantity of young skulls.

Hippopotamus, *Hippopotamus amphibius*
(Laws 1968)

Figure 2. The hippopotamus has largely overcome the hazard of immaturity by the time he is 2 yr old,

with 51% survival from this hazard. A long period of maturity follows, during which there is a constant hazard of 0.039. Of the animals that survive the hazard of immaturity, about one third survive the hazard of adulthood to the age of 30 yr. At this age, the hazard of senescence begins to be felt, with a modal survival time for senescence of 37.5 yr.

TABLE 2. Parameter standard errors.

Animal	μ_0	T_m	$\ell_1(\infty)$	k	T_s	$\hat{\sigma}_s$
Dall Sheep						
10% correction	0.075	0.17		0.014	0.17	0.19
20% corrections	0.090	0.07		0.009	0.12	0.14
Hippopotamus	0.049	0.027		0.0005	0.16	0.20
African Buffalo male			0.004	*	0.13	0.42
Zebra, female	*	*	0.02	0.005	0.31	0.43
Kalo Deer male	*	*	0.064	0.038	*	*
female	*	*		0.012	*	*
Chaparral Deer female	*	*	0.022	0.006	*	*
British Blackbird	*	*	0.10	0.022	*	*
Medaka						
XY	*	*	*	0.0002	1.7	2.6
YY	0.011	2.6		*	1.9	2.7
YY (androgen)	0.0013	1.3		*	3.6	6.0

* Cannot be determined from the data.

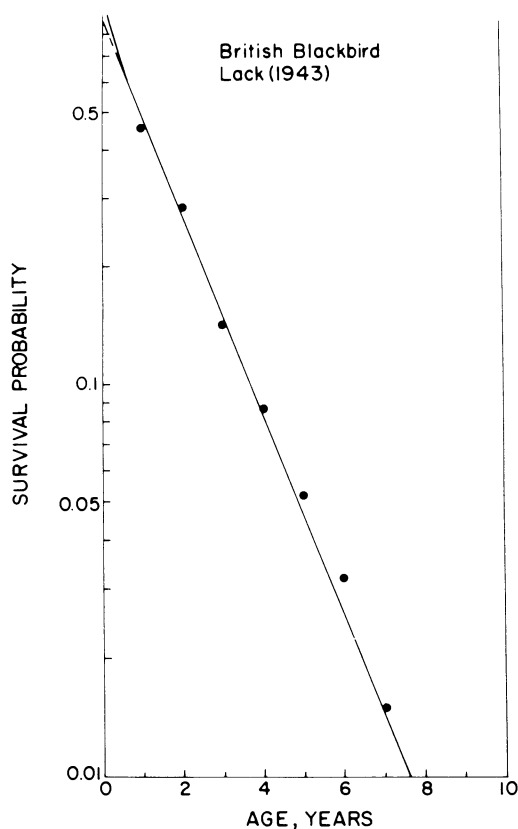


FIG. 7. Proposed mortality model fitted to data for the British Blackbird (*Turdus merula merula*). Similar data and fits occur for nearly all species of small birds for which data from the wild are available.

African Buffalo, Syncerus caffer
(Spinage 1972)

Figure 3. While it is possible to calculate the initial hazard and immaturity time constant, the parameter standard errors are too large to make these parameter values meaningful. One can state confidently that the time constant for immaturity is <1 yr. The survivorship from the hazard of immaturity can be calculated with reasonable confidence as about 50%. A period of 5 or 6 yr of maturity follows during which very few deaths occur, the hazard of adulthood being negligible. The hazard of senescence begins to be felt at an age of ≈ 8 yr, with a modal survival time for senescence being ≈ 12 yr.

Zebra, female, Equus burchelli boehmi
(Spinage 1972)

Figure 4. It is not possible to calculate confidently values for the initial hazard and immaturity time constant for the zebra from the data, except that the value of the immaturity time constant is <1 yr. During the period of adulthood there is a constant hazard of 0.076, sufficiently large so that few animals survive to meet

the hazard of senescence. At an age of ≈ 15 yr, the hazard of senescence begins to appear, with a modal survival time from this hazard being ≈ 17 yr.

Roe Deer of the Kalo, Capreolus capreolus
(Andersen 1953)

Figure 5. These animals, though also wild ungulates, present quite a different picture from the Dall Mountain Sheep. The data do not permit evaluation of the maturity time constant, except that it is <1 yr. Survivorship from the hazard of immaturity is 46% for males and 62% for females; the constant hazard of maturity is less for females, being ≈ 0.5 for males and 0.4 for females.

Because of a very large constant hazard of maturity, probably predation or starvation, survivorship to senescence is too small to estimate the parameters of the senescence hazard; apparently virtually no animals survived long enough to die of the hazards associated with senescence. It seems likely that the modal survival time from senescence would be ≥ 10 yr, consistent with that of the Dall Sheep.

Black-tailed Deer of the chaparral, Odocoileus hemionus columbianus (Taber and Dasmann 1969)

Figure 6. The survival curve for this species and environment is similar to that of the Kalo Deer, but reflects a much less severe environment, or conceivably greater hardiness as a species. Survivorship from the hazard of immaturity is $\approx 80\%$; the constant hazard of adulthood is only 0.15, compared with 62% and 0.4 for the female Kalo Deer.

Unfortunately, although survivorship to 9 yr of age is very appreciable (20%) and there is a hint in the data that the hazard of senescence is beginning to appear at that age, the data do not extend beyond this point. Consequently, the senescence hazard parameters cannot be evaluated, although the modal survival time from senescence is certainly >10 yr.

British Blackbird, Turdus merula merula
(Lack 1943)

Figure 7. A large survivorship is seen from the immaturity hazard (88%); this is probably an overestimate, because a portion of the early time period was not included in the data. The data for this bird indicate a severe constant hazard of nearly 0.60, probably due to predation. As in the case of the Kalo Deer, virtually none survives to die of the senescence hazard.

This picture seems to be typical of small birds; for nearly all, after a period of juvenile adjustment of a year or less, no further adjustment takes place. The senescence hazard cannot be determined in the presence of such heavy predation. Aviary data would be required to determine the senescence hazard. It appears that, in spite of the very short mean life span in the wild, the life span from the senescence hazard alone would be many years.

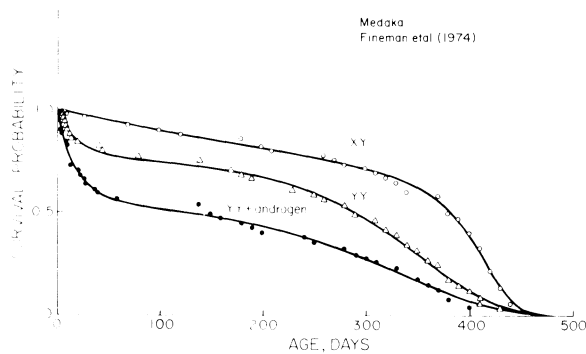


FIG. 8. Proposed mortality model fitted to laboratory survival curve data for the Asiatic killifish, the Medaka (*Oryzias latipes*).

Medaka, Oryzias latipes
(Fineman et al. 1974)

Figure 8. These data on the tropical fish *Oryzias latipes* are of interest in giving a picture of life under laboratory conditions. The data also give a fairly detailed picture of the effect of the hazard of immaturity, thus giving a check on the new distribution here proposed for that hazard. Because these animals (like many fish) can undergo sex reversal, it is possible reliably to obtain YY genotypes; the data here shown are for XY males, YY males, and YY males to which exogenous androgen (methyl testosterone) was administered in food.

Considering the small cohort size (40–50 animals) the fit of the model to the data is encouraging, especially the good fit during the early time period. Because this is the period during which the novel part of the model dominates, it is the period which requires the most careful checking.

The obvious differences between the curves are largely ascribable to the hazard of immaturity. If the animals are arranged in order of "maleness," i.e., XY, YY, YY + androgen, the initial hazard and death from immaturity increase; the constant hazard, never large, drops; the modal life span from the senescence hazard decreases and its variance increases, indicating an increase in the initial hazard of senescence, but a decrease in the rate at which this hazard is accelerated. Fineman et al. (1974) discussed in more detail the effects of genotype, phenotype, and sex on the parameters.

The division of causes for mortality into three types, depending on the adaptive response of the animal to the hazard, appears to have some general validity, at least as far as vertebrates are concerned. The new distribution here proposed for hazards to which the animal adapts, particularly the hazard of immaturity, has some of the characteristics of the Gompertz distribution in inverted form and can be derived from similar reasoning. The finite survivorship from this hazard is biologically sensible. The concurrence of the

three hazard types with Pearl and Miner's (1935) three types of survival curves is also encouraging. In short, the approach appears fruitful, but the real test must await analysis of more life tables to see if increased biological understanding can result from use of the model.

ACKNOWLEDGMENTS

The author is indebted to Dr. George Williams and Dr. John Emlen for making available their results, and to Dr. Max Woodbury for suggesting the Appendix.

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APPENDIX

The formulas given in the body of this paper for relating the Gompertz hazard parameters to survival mean and variance are simple and sufficiently accurate for obtaining initial estimates of parameter values for nonlinear model fitting. Somewhat more accurate values (for the Gompertz parameter ratio $a/b \ll 1$) may be derived by using a cumulant-generating approach (Kendall and Stuart 1969). In this method, the Laplace transform of the senescence survival time density function is first obtained. The logarithm of this transform is differentiated twice with respect to the Laplace variables. To obtain the mean, the Laplace variable in the negative first derivative is set to zero; for the variance, the Laplace variable in the second derivative is set to zero.

As given in the body of this paper, the survival time density function for the Gompertz hazard is:

$$f(t) = a \exp(bt) \exp\{(a/b)[1 - \exp(bt)]\}. \quad (1a)$$

Using the Laplace transform as a moment-generating function, we have

$$M(s) = \int_{t=0}^{\infty} \exp(-st) a \exp(bt) \exp\{(a/b)[1 - \exp(bt)]\} dt. \quad (2a)$$

Rearranging terms, and letting

$$v = e^t, \quad (3a)$$

we have

$$M(s) = a \exp(a/b) \int_{v=1}^{\infty} v^{-s} v^{b-1} \exp[-(a/b)v^b] dv. \quad (4a)$$

A further simplification is achieved by letting

$$x = (a/b)v^b, \quad (5a)$$

which leads to

$$M(s) = \exp(a/b)(a/b)^{s/b} \int_{x=a/b}^{\infty} e^{-x} x^{-s/b} dx. \quad (6a)$$

Noting that by definition

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, \quad (7a)$$

the moment-generating function becomes

$$M(s) = \exp(a/b)(a/b)^{s/b} \cdot \left[\Gamma(1 - s/b) - \int_{x=a/b}^{a/b} e^{-x} x^{-s/b} dx \right]. \quad (8a)$$

Assuming that $a/b \ll 1$, we neglect the integral term in Eq. 8a, giving

$$M(s) = \exp(a/b)(a/b)^{s/b} \Gamma(1 - s/b). \quad (9a)$$

Taking the natural logarithm of $M(s)$, we have

$$L = L[M(s)] = a/b + (s/b) \ln(a/b) + \ln[\Gamma(1 - s/b)]. \quad (10a)$$

To obtain an estimate of the mean, we differentiate L once with respect to s , change sign, and then set $s = 0$, yielding

$$\frac{dL}{ds} = (1/b) \ln(b/a) - \frac{d}{ds} [\ln \Gamma(1 - s/b)]. \quad (11a)$$

By definition

$$\frac{d}{dz} [\ln \Gamma(z)] = \Psi(z), \quad (12a)$$

where $\Psi(z)$ is the digamma function.

To utilize this relation, we let

$$1 - s/b = u, \quad ds = -bdu, \quad (13a)$$

so that

$$\frac{dL}{ds} = (1/b) \ln(b/a) + (1/b) \Psi(1 - s/b). \quad (14a)$$

By setting $s = 0$ in Eq. 14a, we obtain an approximation to the survival time for $a/b \ll 1$, or

$$\mu \doteq (1/b) \ln(b/a) + (1/b) \Psi(1),$$

or

$$\mu \doteq (1/b) \ln(b/a) - 0.57721/b. \quad (15a)$$

To obtain an estimate for the variance, we differentiate Eq. 14a again with respect to s :

$$\frac{d^2 \ln L}{ds^2} = (1/b^2) \Psi'(1 - s/b) \quad (16a)$$

in which $\Psi'(z)$ is the trigamma function. An estimate for the survival time variance is obtained by setting s to zero in Eq. 16a:

$$\sigma^2 = (1/b^2) \Psi'(1). \quad (17a)$$

The standard deviation is thus obtained:

$$\sigma = \sqrt{\Psi'(1)} / b = 1.283/b. \quad (18a)$$