AGE STRUCTURE, GROWTH, ATTRITION, AND ACCESSION: A NEW SYNTHESIS

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This paper shows that each of the equations describing relationships among demographic parameters in a stable population is a special case of a similar and equally simple equation that applies to any closed population. An equation almost as simple applies to <u>any</u> population defined in most general terms as a collectivity classified by an index analogous to age. The paper then demonstrates some implications of these new equations for demographic theory and practice.

Our work on this subject has precursors in the efforts of Von Foerster (1959), Trucco (1965), Langhaar (1972), Hoppensteadt (1975), and Bennett and Horiuchi (1981). In particular, these works recognize that there is a necessary relationship in a closed population between a population's age distribution at time t, its age-specific force of mortality function at time t, and its set of age-specific growth rates at time t. From this recognition, we take the short step required to rewrite the mathematics applying to stable populations in a more general form.

The extension to more general conditions of the relations found in stationary and stable populations can be understood by considering the expression for the relative rate of change of the number of persons at each age as age advances. If the number of persons in a population is assumed to be a continuous function of age, then the relative change in number as age increases is

$$\frac{1}{N(a)}\frac{dN(a)}{da}, \quad \text{or} \quad \frac{d \log N(a)}{da} .$$

Here N(a) refers to N(a,t), the number of persons aged a at time t; we have

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omitted the t identifier for convenience. A stationary population is a population with the same number of births every year and an unchanging schedule of mortality rates. In a stationary population, the number of persons at each age does not change with time. In such a population

$$\frac{1}{N(a)}\frac{dN(a)}{da} = -\mu(a)$$

where $\mu(a)$ is the age-specific mortality rate (or force of mortality) at exact age a.

A stable population is a population in which the number of births changes with time at a constant rate r, and the mortality schedule is the same from year to year. The number of persons at each individual age also changes with time at the rate r. As a result, each successively younger cohort is larger (or smaller if r is negative) at every age than its older predecessor by a constant multiple. If we imagine a growing stable population in which there is no mortality, the relative number at age a would diminish at a rate r, or

$$\frac{1}{N(a)}\frac{dN(a)}{da} = -r \quad .$$

Since the stable population is in general subject to a fixed mortality schedule $\mu(a)$, the relative number changes with age as the result of the independent effects of mortality at age a and the relative difference in size of adjacent cohorts r, or

$$\frac{1}{N(a)}\frac{dN(a)}{da} = -\mu(a) - r \qquad (1)$$

as can be verified by differentiating the well-known expression for the age distribution of a stable population ($N(a) = Be^{-ra}p(a)$).

The extension to less restricted conditions in which mortality and fertility change with time is simple. In any closed population, the relative number at age a changes as age advances because of mortality; it also changes as a large or small cohort advances in age, replacing one different in size. To make equation (1) applicable to any closed population at any moment in time, one can express the rate of increase in the number at age a as a function of age. Thus, at any moment

$$\frac{1}{N(a)}\frac{dN(a)}{da} = -\mu(a) - r(a) \qquad (2)$$

when $r(a)$ is defined as

$$\lim_{\Delta t \to 0} \frac{N(a,\ t + \Delta t) - N(a,\ t)}{N(a,t)\Delta t}$$

The validity of equation (2) can be justified intuitively by noting that the number at a slightly greater age than a at time t, or $N(a + \Delta a,\ t)$, equals the number at age a at a slightly earlier time, or $N(a,\ t-\Delta t)$, less the number of deaths the cohort has experienced in this short period (note that Δt is necessarily equal to Δa). The number of deaths is $N(a,t)\ \mu(a,t)\ \Delta t$, if the effect of the difference in cohort size on the number of deaths is ignored, as it may be as $\Delta t(=\Delta a)$ approaches zero. Hence

$$\frac{N(a + \Delta a,t) - N(a,t)}{N(a,t)\Delta a} = \frac{N(a,t - \Delta t) - N(a,t)}{N(a,t)\Delta t} - \frac{\mu(a,t)\Delta t}{\Delta a};$$

the limit of this expression as Δa (= Δt) approaches zero is equation (2). More simply, equation (2) expresses the relative change in numbers with age as the sum of two independent terms, the change that would occur as the result of mortality alone, and the change that would occur as the number at age a changed with time, in the absence of mortality.

Since (2) can be written as $\frac{d \log N(a)}{da} = -\mu(a) - r(a)$, it follows by integration that

$$N(a) = N(o)e^{-\int_o^a r(x)dx - \int_o^a \mu(x)dx}, \text{ or}$$

$$N(a) = Be^{-\int_o^a r(x)dx}\ p(a). \qquad (3)$$

Equation (3) is the basis of much of the rest of this paper. So that the elements of this equation are as clearly understood as possible, let us reiterate that

$N(a)$ = number of persons age a at time t, i.e., height of the $N(a,t)$ surface at some point a at some time t.

$p(a)$ = probability of surviving from age 0 to age a according to the life table prevailing at time t, or $p(a) = e^{-\int_o^a \mu(x)dx}$ where $\mu(x)$ is the mortality function at time t.

$r(x)$ = annual growth rate of persons aged x evaluated at time t.

Unless otherwise noted, all functions in this paper pertain to some particular time t; all relations among functions pertain at each and every time t.

It seems likely that equation (3) has been derived many times in many different contexts. But its implications for demographic analysis do not appear to have been fully developed. Part of the neglect may result from the belief that the r(x) series is theoretically uninteresting, since it is clearly a function of past patterns of mortality and fertility. But to a demographer, the r(x) series is a very widely observed datum, calculable whenever a country has taken two censuses not too widely separated. With that datum, many relations among other demographic parameters can be clarified. We will now show how it leads to a simple generalization of the equations characteristic of a stable population.

The birth rate of the population is

$$b = \frac{B}{\int_0^\infty N(a)da} = \frac{B}{\int_0^\infty Be^{-\int_0^a r(x)dx} p(a)da} = \frac{1}{\int_0^\infty e^{-\int_0^a r(x)dx} p(a)da} \tag{4}$$

The proportion of the population that is age _a_ is

$$c(a) = \frac{N(a)}{\int_0^\infty N(a)da} = \frac{Be^{-\int_0^a r(x)dx} p(a)}{\int_0^\infty Be^{-\int_0^a r(x)dx} p(a)da}, \text{ or}$$

$$c(a) = be^{-\int_0^a r(x)dx} p(a) . \tag{5}$$

Finally, the birth rate can also be represented as $b = \int_\alpha^\beta c(a)m(a)da$, where $m(a)$ is the rate of bearing female children for women aged _a_ and α and β are the lower and upper ages of childbearing. Substituting (5) into this last equation, we have

$$b = \int_\alpha^\beta be^{-\int_0^a r(x)dx} p(a)m(a)da, \text{ or}$$

$$1 = \int_\alpha^\beta e^{-\int_0^a r(x)dx} p(a)m(a)da \quad . \tag{6}$$

If age-specific growth rates are constant with age at a value of _r_, equations (4), (5), and (6) become

$$b = \frac{1}{\int_0^\infty e^{-ra} p(a)da} \tag{4'}$$

$$c(a) = be^{-ra} p(a) \tag{5'}$$

$$1 = \int_\alpha^\beta e^{-ra} p(a)m(a)da \quad . \tag{6'}$$

Equations 4', 5', and 6' are readily seen to be the classic equations characterizing stable populations (Lotka, 1939; Coale, 1972). Thus, the stable equations are a special case of a more general set of equations 4 - 6; the stable equations pertain whenever age-specific growth rates are constant. Equations 4 - 6 characterize every closed population at every moment in time.

The existence of a set of such simple and general relations, in view of the large volume of work on stable population theory, is surprising.

The development so far has assumed the population to be closed to migration. However, the formulation can be immediately generalized to an open population with an age-specific force of net out-migration function of e(x). It is only necessary to recognize that the force of migration function acts on the growth process in a fashion exactly analogous to the action of mortality. The age distribution does not recognize whether people are leaving the population by death or by out-migrating, and net in-migration will simply offset (sometimes more than completely) the impact of mortality. As shown in the Appendix,

$$N(a) = N(o)e^{-\int_o^a r(x)dx} e^{-\int_o^a e(x)dx} p(a) \qquad (7)$$

The three basic equations 4-6 can now be derived as from (3) above, simply by adding e(x) to r(x). With this correction for migration, any open population can be analytically converted into a closed one.

In fact, nothing limits us to recognizing only one form of "migration" or even one form of mortality. Any form of attrition or accession can be introduced into (7) simply by recognizing that it must act analogously to migration or mortality from all combined causes. Equation (7) is the basis of a surprisingly general set of relations. In particular, one can see that the age composition of any population at any moment (assuming only that age composition and its change through time are continuous) is completely determined by the rate of increase in the number at each age at the given moment, together with the rate of attrition (including negative attrition) at each age from each of a number of independently operating factors. To be more specific, if the rate of increase, r(x), is known for each age _x_ from zero to the highest age attained, and if the values of _i_ different attrition factors, $\mu_i(x)$, are also known, the age composition is completely determined and can readily be calculated; conversely, if the age distribution and all but one of the attrition factors are known, the rate of attrition for the omitted factor can readily be calculated.

This set of relations is known in demography, for particular instances, and the basic equation in differential form is familiar in mathematical biology and actuarial work, but the full (though simple) generalization seems to have escaped attention. The basic equation is as follows:

$$N(a,t) = N(o,t)e^{-\int_o^a r(x,t)dx} e^{-\sum_i \int_o^a \mu_i(x,t)dx} ,$$

where N(a,t) is the population density at age _a_, time _t_; r(x,t) is the instantaneous rate of growth of the population at age _x_, time _t_; and $\mu_i(x,t)$ is the rate of attrition from the operation of the i^{th} among the several factors that diminish (or increase; the attrition can be negative) the number of members of the population at age x. Because all of the variables are defined at the same moment (t), the time variable can be suppressed, and the equation expressed as:

$$N(a) = N(o)e^{-\int_o^a r(x)dx} e^{-\sum_i \int_o^a \mu_i(x)dx} \qquad (8)$$

Note how wide is the universe to which the equations apply. To be consistent with these equations, members of a collectivity must have a defined duration of existence in a given state, a defined duration analogous to age. Conventional chronological age of humans is duration of life since birth, but duration of marriage, duration of residence, duration of existence in the single state, and duration of stay in a hospital are other examples from human experience. The attrition factors -- mortality, or mortality from each of several independent causes, out-migration (or immigration, which is negative out-migration), divorce (attrition from the married state), or marriage (attrition from the single state or negative attrition into the married state) -- cause a specified proportionate rate of decline (or increase) in numbers at each age for a defined collectivity. For the relation to hold, the distribution of numbers and the force of each attrition factor must be continuous functions of age.

Although r(x) is formally defined as $\frac{\partial N(a,t)}{N(a,t)\partial t}$, it can be viewed and manipulated as a function of age, and not of time at a given moment. An analog is the speed of an automobile, which is properly defined as the time derivative of the automobile's position, but can also be viewed as a characteristic of the vehicle at a given moment, indicated by the speedometer reading. A speed of 60 miles per hour has no implication that the car will cover 60 miles in an hour nor that it has covered 60 miles in the past hour. The speedometer is usually a voltmeter showing the voltage produced by a generator mounted on the driveshaft, a generator producing a voltage (ideally) proportional to the rate of rotation of the shaft. One can imagine a "speedometer" that reads r(x) at each moment in a given population. In fact, if the attrition factors and the age distribution in equation (8) are known, r(x) can be calculated without any record of the change in number at the same age from one moment to the next. Note further that any of the age functions in equation (8) -- r(x), $\mu_i(x)$, or N(a) -- can be calculated from a full listing of all of the others.

In equation (8) r(x) is formally analogous to any one of the i attrition factors. Mathematically, it could be included as the $(i+1)^{th}$ form of attrition: a population subject to no external attrition factors decreases with age to a degree that is proportional to the rate of increase at each age. However, the rate of increase is distinctive in that it is a built-in form of attrition, the result of differences in cohort size that in turn arise from the past history of the population -- from past rates of entry and attrition -- whereas the other sources of "attrition" are exogenous.

Any population can be thought of as a stationary population subject to multiple "decrements", one of which is growth. As in the conventional multiple decrement situation, it is possible to ask what the population structure would be like if one of the decrements were not operating. If the eliminated "decrement" is growth, we are left with the stationary population produced by the activity of the exogenous decrements , $\mu_i(x)$. If mortality is the only remaining source of decrement, the stationary population is the conventional stationary population of life table literature. In other words, to convert the age distribution at time t into the age distribution of a hypothetical stationary population subject to current forces of attrition and a radix of today's births, it is only necessary to multiply the current number of persons aged a by $\exp\{ \int_o^a r(x,t)\, dx \}$. This conversion factor appears in virtually every formula in this paper because it transforms any population into its corresponding stationary population, from which many demographic functions can be derived.

The Age Distribution of Births and Deaths

The frequency distribution of mothers' ages at childbearing at time t is

$$v(a) = \frac{N(a)m(a)}{\int_\alpha^\beta N(a)m(a)da} = \frac{Be^{-\int_o^a r(x)dx} p(a)m(a)}{\int_\alpha^\beta Be^{-\int_o^a r(x)} p(a)m(a)da}, \text{ or}$$

$$v(a) = e^{-\int_o^a r(x)dx} p(a)m(a) .$$

It is because the term on the right-hand side of this expression is the frequency distribution of mothers' age at childbearing that it must sum to unity, as in equation (6).

An intuitive understanding of this formula may derive from the following considerations. Rewriting the above equation as

$$v(a)e^{\int_o^a r(x)dx} = p(a)m(a),$$

we observe that the right-hand side is the expected number of births at age a per newborn child subject for all her life to today's p(a) and m(a) schedules. The left-hand side consists of two components: B(a)/B, or births occurring today at age a per newborn child; and $\exp\left\{\int_0^a r(x)dx\right\}$, which expresses the factor by which births at age a would grow over the next a years, under current fertility and mortality rates, as persons now aged a are replaced by the larger (or smaller) cohort just now being born. Thus, both sides of the equation are exact representations of the expected number of births a years hence per woman in the cohort just now being born, if she is subject to current p(a) and m(a) schedules.

We may now integrate both sides of this equation to derive a new expression for the net reproduction rate:

$$NRR = \int_\alpha^\beta p(a)m(a)da = \int_\alpha^\beta v(a)e^{\int_0^a r(x)dx} da \quad . \qquad (9)$$

This expression says that the net reproduction rate in any closed population can be estimated exactly from information on the distribution of mothers' ages at childbirth and from age-specific growth rates. The corresponding relation in a stable population seems to have escaped comment, probably because the normal analytic problem is to estimate r_I from p(a)m(a) and not the reverse. But if r(x) is observed and v(x) is known or can be approximated, the net reproduction rate can be estimated from the set of growth rates, rather than customary estimation of the intrinsic rate of increase from the net reproduction rate.

The frequency distribution of ages at death in a closed population likewise bears a simple relationship to the corresponding frequency in the underlying life table that is generating the data. As Bennett and Horiuchi (1981) have shown, the number of deaths at age a (time t) is

$$D(a) = N(a)\mu(a) = N(0)e^{-\int_0^a r(x)dx} p(a)\mu(a), \text{ or}$$

$$D(a) = N(0)e^{-\int_0^a r(x)dx} d(a), \text{ where}$$

d(a) = deaths at age a in the life table prevailing at time t (with radix of one).

So the frequency distribution of ages at death is

$$\frac{D(a)}{\int_0^\infty D(a)da} = \frac{d(a)e^{-\int_0^a r(x)dx}}{\int_0^\infty d(a)e^{-\int_0^a r(x)dx} da}$$

Normally, the analytic problem will be to infer life table deaths from the observed age distribution of deaths. For this purpose, one would use

$$\frac{d(a)}{\int_0^\infty d(a)da} = d(a) = \frac{D(a)e^{\int_0^a r(x)dx}}{\int_0^\infty D(a)e^{\int_0^a r(x)dx} da} .$$

From the life table death function, d(a), all other mortality functions of interest can be reconstructed.

Population at Age a Determined by Accessions and Departures at Ages from Zero to a, or from a to ω

This section shows how the number of persons at a particular age is related to the contemporaneous accessions and exits occurring below that age, as well as to accessions and exits above that age. Denote accessions at age x as $A(x)$, the number of exits as $E(x)$, the rate of accession $A(x)/N(x)$ as $\mu^+(x)$, and the rate of exit as $\mu^-(x)$. The rate of increase at x is $r(x)$. If we imagine a hypothetical cohort of $N'(o)$ original members subject to $\mu^-(x)$ and $\mu^+(x)$, then the number at age a, $N'(a)$, would be

$$N'(o)e^{\int_0^a (\mu^+(x) - \mu^-(x))dx} .$$

$A'(x)$ would equal $N'(x)\mu^+(x)$, and $D'(x)$ would equal $N'(x)\mu^-(x)$. In the actual population (assuming $N'(o) = N(o)$), $N(x) = N'(x)e^{-\int_0^x r(y)dy}$; hence $A(x) = A'(x)e^{-\int_0^x r(y)dy}$, and $D(x) = D'(x)e^{-\int_0^x r(y)dy}$. The purpose of defining the number of accessions and departures in a hypothetical cohort is to make use of two identities that apply to a cohort: the number of persons at age a equals the number at zero plus the sum of accessions, less the sum of departures, in the interval from zero to a; the number at a also equals the number of departures less the number of accessions, in the interval from a to the highest age attained, ω, at which age the cohort is extinct.

Thus:

$$N'(a) = N'(o) + \int_0^a (A'(x)-D'(x))dx; \text{ also} \qquad (10)$$

$$N'(a) = \int_a^\omega (D'(x)-A'(x))dx. \qquad (11)$$

Now we recall the relations listed above between numbers at each age, and numbers of accessions and departures, in the actual population, and in the hypothetical cohort. Substituting from the equations in the preceding paragraph for N'(a), A'(x), and D'(x) the corresponding values of N(a), A(x), and D(x) in (10) and (11), we find

$$N(a) = \{N(o) + \int_o^a (A(x)-D(x))e^{-\int_o^x r(y)dy} dx\}e^{-\int_o^a r(x)dx} \text{, or}$$

$$N(a) = N(o)e^{-\int_o^a r(x)dx} + \int_o^a (A(x) - D(x))e^{-\int_x^a r(y)dy} dx,$$

or, counting N(o) as A(o),

$$N(a) = \int_o^a (A(x) - D(x))e^{-\int_x^a r(y)dy} dx \qquad (12)$$

and

$$N(a) = \int_a^\omega (D(x)-A(x))e^{\int_a^x r(y)dy} dx \quad . \qquad (13)$$

These equations can also be expressed in a form that facilitates calculation, namely

$$N(a+n) = N(a)e^{-\int_a^{a+n} r(x)dx} + \int_a^{a+n} (A(x)-D(x))e^{-\int_x^{a+n} r(y)dy} dx \qquad (14)$$

and

$$N(a-n) = N(a)e^{\int_{a-n}^a r(x)dx} + \int_{a-n}^a (D(x)-A(x))e^{\int_{a-n}^x r(y)dy} dx. \qquad (15)$$

As an experiment, these equations were used to calculate the number of currently married women at each age in Sweden in 1976, counting accessions as the number of marriages plus the number of immigrant married women at each age, and departures as emigration of married women, divorce, death, and loss of husband. The only use of data on the number of resident women is to calculate the crucial age-specific growth rates for the married population. The calculated numbers duplicate the recorded number of married women by single years of age with an average error (from age 17 to age 30) of 1.3 percent.

Particular Features of the New Equations

Equations (5) and (6) can be puzzling to anyone habituated to traditional demographic analysis, including those at home with the mathematics of stable populations. Equation (6) presents a relation that must hold between the net maternity function, $p(a)m(a)$, experienced by the populations at a given time, and the set of age-specific growth rates, $r(x)$, found at the same time. Conventionally, the net maternity function is thought of as having implications for growth in the long run, when the "intrinsic" growth rate has time to manifest itself. It is not obvious why (in terms other than found in the formal proof) the set of contemporaneous growth rates must also necessarily be consistent with the net fertility function. The puzzle is solved by recognizing that $r(x)$ for all ages above zero is, as common sense suggests, causally independent of the net fertility function of the moment, but not of the growth rate at age zero. If the net fertility function is changing from year to year because of changes in the rate of childbearing, it is the role of the growth rate in the neighborhood of age zero to be modified in such a way as to ensure that equation (6) continues to hold.

This outcome can be clarified by separating the integral $\int_o^a r(x)dx$ in equation (6) into $\int_o^1 r(x)dx + \int_1^a r(x)dx$, a separation that is permissible because the range of <u>a</u> begins at α, well above age 1. $\int_o^1 r(x)dx$ is part of $\int_o^a r(x)dx$ for all relevant <u>a</u>. It follows that

$$e^{-\int_o^1 r(x)dx}$$

can be factored from equation (6) as follows:

$$\int_\alpha^\beta e^{-\int_o^a r(x)dx} p(a)m(a)da = \int_\alpha^\beta e^{-\int_o^1 r(x)dx} e^{-\int_1^a r(x)dx} p(a)m(a)da =$$

$$e^{-\int_o^1 r(x)dx} \int_a^\beta e^{-\int_1^a r(x)dx} p(a)m(a)da.$$

If we call $\int_o^1 r(x)dx$ ${}_1r_o$, it follows from equation (6) and this decomposition that

$${}_1r_o = \ell n \int_a^\beta e^{-\int_1^a r(x)dx} p(a)m(a)da \ . \qquad (6a)$$

Thus, ${}_1r_o$ has a determinate form that depends on the net fertility function and $r(x)$ from $x=1$ to β. In a stable context, of course, all of the values of $r(x)$ above age one are the same, and ${}_1r_o$ will be found to have this value as well. If the net reproduction rate of a formerly stable population is reduced by 50 percent in one year, the value of

$$\ln \int_{a}^{\beta} e^{-\int_{1}^{a} r(x)dx} p(a)m(a)$$

will be approximately $\ln(\frac{1}{2})$; ${}_1r_0$ will be approximately $\ln(\frac{1}{2})$; and $e^{-{}_1r_o}$ will be about two, maintaining the validity of equation (6). In short, it follows from equation (6a) that each year the growth rate at age zero, being fully determined by the growth rates of older cohorts and the current net fertility schedule (no matter how aberrant), maintains the consistency of the full set of growth rates with net fertility.

The connection between current growth rates and the intrinsic growth rate corresponding to the p(a) and m(a) schedules can be seen by rewriting equation (6) as

$$\int_{\alpha}^{\beta} e^{-(\bar{r}_a - r_I)a} e^{-r_I a} p(a)m(a)da = 1 \quad .$$

We have denoted $\int_{o}^{a} r(x)dx/a$ as $\bar{r}_a$, the mean of age-specific growth rates below age _a_ in the population; r_I is the intrinsic rate. Since $e^{-r_I a}$ p(a)m(a) is the frequency distribution of ages at childbirth in the stable population, it simply acts as a set of weights applied to the exp $\{-(\bar{r}_a - r_I)a\}$ schedule. The weighted sum of this latter schedule must be unity; therefore, $\bar{r}_a$ cannot lie perpetually above (or below) r_I in the childbearing interval. The two values must be equal for at least one age between α and β. Thus, the intrinsic growth rate in any closed population must equal the average current age-specific growth rate below some age that lies within the childbearing interval. In Japan, the intrinsic growth rate for 1960-1964 was -.0033, which equals the mean age-specific growth rate during the 1960-1963 period below age 29.26.[1]

Of the new expressions, (5) is perhaps the most puzzling. Why should the proportion of the population that is aged _a_ at time _t_ be a simple function of the birth rate at _t_, the life table at _t_, and age-specific growth rates at _t_? It seems intuitively compelling that information on the history of birth and death rates would have to be introduced in order to determine the value of c(a). But in this case, all of the pertinent history is contained in the contemporaneous age-specific growth rate function.

To gain a better idea of the basis of (5), first imagine that mortality is constant. The size of the cohort of births in year _t_ relative to the size of population is, by definition, b(t). With constant mortality, however, the only possible source of age-specific growth is growth in the numbers entering successive birth cohorts. So the number of births _a_ years earlier must have been smaller (or larger) than the number at t by the factor $\exp\{-\int_0^a r(x)dx\}$. Thus, the size of the cohort of births born at time (t-a), relative to the size of population at time t, is $b \cdot \exp\{-\int_0^a r(x)dx\}$. However, only the fraction p(a) from that cohort born a years earlier has survived, so that the proportion of the population now aged a is $b \cdot \exp\{-\int_0^a r(x)dx\}\, p(a)$. The basis for (5) is thus clear when mortality is constant.

To generalize this result to the case of changing mortality, suppose that mortality among the cohort now aged a was higher than that pertaining at time t by amount $\Delta u(j)$ at age $j<a$. Then for the cohort, $p_c(a) = p(a, t)e^{-\Delta u(j)}$. But if mortality was higher by $\Delta u(j)$ at time t-j, then its subsequent reduction must have raised the growth rate by $\Delta u(j)$ at some age between j and a at time t, relative to the growth rate under constant mortality conditions. A gradual reduction of $\Delta u(j)$ would spread the growth boost over several ages by correspondingly smaller amounts. Which age received the growth boost is immaterial; what matters is that r(x) has risen by $\Delta u(j)$ at some age below a, so that the series $\exp\{-\int_0^a r(x)dx\}$ is changed by the factor $\exp\{\Delta u(j)\}$. This factor exactly offsets the effect of the altered mortality history for the cohort aged a, and the expression for c(a) is unaltered. Simply stated, any difference between the mortality history of a cohort and the current mortality regime will be completely reflected in the r(x) series. Likewise, any growth in the number of births will also be reflected completely in r(x). That is why no "history" is required in equation (5).

The connection between the equations and a population's history can be made more explicit by recognizing that there are two expressions for N(a,t) in a closed population. From (3) we have

$$N(a,t) = N(0,t)\, e^{-\int_0^a r(x,t)}\, p(a,t).$$

But by definition the number of persons aged a at time t is equal to births that occurred a years earlier times the proportion of that birth cohort who survived to age a, $p_c(a)$. Therefore,

$$N(a,t) = N(0,t-a)\, p_c(a).$$

Combining these two expressions for N(a,t) gives

$$\int_0^a r(x, t) = \bar{r}_B \cdot a + \int_0^a \Delta\mu(x)dx, \text{ where}$$

$\bar{r}_B$ is the mean growth rate in number of births between time t-a and t; and $\Delta\mu(x)$ is the difference between the cohort and the period death rate at age x, i.e., $\mu(x,t-a+x) - \mu(x, t)$. Thus, the sum of period age-specific growth rates up to age a, time t reflects both the growth rate of entrants to cohorts over the previous a years and any changes in age-specific mortality that have occurred since a particular age was achieved by the cohort now aged a.[2] Arthur (1981) has explored stable population theory using cohort-specific mortality functions.

Illustrative Applications to Sweden

This section demonstrates empirically that with accurate demographic statistics it is possible to use the relations developed above to derive one demographic series -- in this case the age distribution -- from knowledge of certain other series. First, it will be shown that the basic equations can be extended to populations living through a time interval rather than defined at a moment, and to grouped age distributions rather than the population density at age a. Equation (2) is also valid if N(a) is defined as the number of persons reaching age a during a time period T (extending from t' to t"), rather than as the density of population at age a at a given moment. In this case r(a) is $\lim_{\Delta t \to 0} \frac{N(a+\Delta t)-N(a)}{N(a)\Delta t}$, where $N(a+\Delta t)$ is the number arriving at age a during the time interval $t'+\Delta t$ to $t''+\Delta t$. ($\mu(a)$ is defined as the limit, as Δa approaches zero, of the ratio of deaths to persons at ages a to $a+\Delta a$ to person-years lived at these ages, during the period T.) Note that r(a) is $\frac{N(a,t'') - N(a,t')}{N(a)}$, which equals $\log (N(a,t'')/N(a,t'))/T$, the conventional basis for calculating r(a) during a period, if growth in the number reaching age a is constant during T. Equation (2) is extended to a population defined in finite age intervals as follows. Let ${}_nN_{x,t}$ be the number of persons at ages x to x + n at time t.

$$ {}_nN_{x+\Delta x,t} = {}_nN_{x,t-\Delta t} - ({}_nN_x)({}_nM_x)(\Delta t) $$

where ${}_nM_x$ is the death rate from x to x+n. Subtracting ${}_nN_{x,t}$ from both sides of this equation, dividing by $({}_nN_{x,t}).(\Delta x)$, and letting Δx approach zero, we find

$$ \frac{d \log {}_nN_x}{dx} = -{}_nr_x - {}_nM_x , $$

where ${}_nr_x$ is the rate of increase of the population in the age interval x to x+n. From integration and exponentiation of both sides, it follows that:

$$ {}_nN_x = {}_nN_o \, e^{-\int_o^x {}_nr_y \, dy - \int_o^x {}_nM_y \, dy} . $$

Since in a stationary population $\frac{d \log {}_nL_x}{dx} = -{}_nm_x$, it follows that

$$ e^{-\int_o^x {}_nM_y \, dy} \doteq {}_nL_x/{}_nL_o . $$

Thus this equation can be written as

$$ {}_nN_x = {}_nN_o \, e^{-\int_o^x {}_nr_y \, dy} \, {}_nL_x/{}_nL_o \qquad (3a) $$

By an extension of the argument in the first part of this section, it is clear that equation (3a) applies to the distribution of person-years lived during a time interval. The derivations of equations (3) and (3a) are repeated, in terms of differential and integral calculus of functions of two variables, in the Appendix.

The following illustrative calculations are made in this section:

1) The single-year age distribution of the mean population of Swedish females in 1976 is calculated from the number of female births in 1976, the single-year female life table for 1976, the rate of increase in 1976 of females in each single-year age interval, and the rate of net migration at each age. The equation involved is

$$ N(a) = N(o)e^{-\int_o^a r(x)dx \, - \, \int_o^a e(x)dx} \, p(a), $$

where e(x) is the rate of net out-migration at age x. Since the data are available at one-year age intervals, this equation is approximated by

$$ {}_1N_a = B \, e^{-\sum_o^{a+.5} \{{}_1r_x + {}_1e_x\}} \, {}_1L_a/\ell_0 $$

$$ \text{where} \quad \sum_o^{a+.5} {}_1r_x \ \text{is} \ {}_1r_o + {}_1r_1 + \ldots \, {}_1r_{a-1} + {}_1r_a/2. $$

Results are shown in Table 1.

2) The single-year age distribution of the mean population of Swedish females in 1976 is calculated from the 1976 growth rate, the number of female deaths in 1976, and the number of female net out-migrants at each age in 1976. The equation is:

$$ N(a) = \int_a^{\omega} \{D(x)+E(x)\} e^{\int_a^x r(y)dy} \, dx. $$

With data by single-year intervals, this equation was approximated by an iterative calculation:

$$ N(a) = N(a+1)e^{{}_1r_a} + (D(a)+E(a))e^{{}_1r_a/2}. $$

${}_1N_a$ was calculated as $\sqrt{N(a)\cdot N(a+1)}$. Since growth rates above 100 can be determined only for the population above 100 as a whole, while deaths by

Table 1: Number of females in Sweden in 1976, by single years of age, calculated from $N(a) = Be^{-\int_0^a (r(x) + e(x))dx} p(a)$, compared with recorded mean population.

Age	Rate of Increase in 1976 r(x)	Rate of Out-Migration e(x)	$e^{-\int_0^a (r(x)+c(x))dx}$	$_1L_a/l_o$	Estimated Population	Recorded Population	Estimated -Recorded	Proportionate Error
0- 1	-.04894	-.00936	1.02958	0.99325	48871.	49054.	-183.	-0.00373
1- 2	-.05537	-.01096	1.09577	0.99226	51962.	52371.	-409.	-0.00782
2- 3	0.01052	-.00646	1.13042	0.99188	53584.	53901.	-317.	-0.00588
3- 4	-.01486	-.00418	1.13892	0.99165	53975.	54255.	-280.	-0.00517
4- 5	-.00692	-.00351	1.15583	0.99141	54763.	55033.	-270.	-0.00491
5- 6	0.03749	-.00421	1.14269	0.99109	54123.	54420.	-297.	-0.00546
6- 7	0.02390	-.00338	1.11237	0.99083	52673.	52940.	-267.	-0.00505
7- 8	-.05674	-.00178	1.13371	0.99062	53672.	53928.	-256.	-0.00475
8- 9	-.05990	-.00220	1.20419	0.99043	56997.	57296.	-299.	-0.00521
9- 10	-.00924	-.00202	1.24918	0.99021	59114.	59401.	-287.	-0.00483
10- 11	-.00244	-.00232	1.25922	0.98999	59576.	59880.	-304.	-0.00507
11- 12	-.00301	-.00161	1.26515	0.98979	59844.	60127.	-283.	-0.00470
12- 13	0.08868	-.00180	1.21417	0.98957	57420.	57750.	-330.	-0.00571
13- 14	0.04765	-.00157	1.13608	0.98942	53719.	53987.	-268.	-0.00497
14- 15	0.02742	-.00148	1.09590	0.98920	51808.	52049.	-241.	-0.00464
15- 16	0.01900	-.00210	1.07267	0.98884	50691.	50943.	-252.	-0.00494
16- 17	-.01336	-.00383	1.07283	0.98845	50678.	50974.	-296.	-0.00580
17- 18	-.00310	-.00503	1.08649	0.98806	51304.	52144.	-840.	-0.01612
18- 19	-.02129	-.00658	1.10623	0.98757	52210.	52604.	-394.	-0.00749
19- 20	0.00710	-.01036	1.12358	0.98705	53000.	53499.	-499.	-0.00932
20- 21	0.00026	-.01071	1.13130	0.98662	53342.	53861.	-519.	-0.00964
21- 22	0.02117	-.00994	1.13087	0.98613	53295.	53802.	-507.	-0.00943
22- 23	-.03333	-.00972	1.14900	0.98551	54115.	54640.	-525.	-0.00961
23- 24	-.00080	-.00880	1.17965	0.98494	55527.	56035.	-508.	-0.00907
24- 25	-.00135	-.00670	1.19010	0.98448	55993.	56452.	-459.	-0.00814
25- 26	-.04017	-.00547	1.22248	0.98405	57490.	57943.	-453.	-0.00781
26- 27	-.05212	-.00445	1.28658	0.98356	60475.	60927.	-452.	-0.00743
27- 28	-.04056	-.00308	1.35268	0.98300	63546.	63666.	-120.	-0.00188
28- 29	-.00782	-.00195	1.38929	0.98246	65230.	65598.	-368.	-0.00561
29- 30	-.01659	-.00194	1.40909	0.98189	66121.	66494.	-373.	-0.00561
30- 31	0.00896	-.00180	1.41712	0.98124	66454.	66816.	-362.	-0.00542
31- 32	0.01496	-.00166	1.40269	0.98060	65734.	66090.	-356.	-0.00538
32- 33	0.05771	-.00154	1.35480	0.97995	63448.	63805.	-357.	-0.00560
33- 34	0.09237	-.00123	1.25860	0.97919	58897.	59249.	-352.	-0.00594
34- 35	0.11138	-.00106	1.13800	0.97832	53206.	53547.	-341.	-0.00637
35- 36	0.05332	-.00144	1.04935	0.97737	49014.	49278.	-264.	-0.00537
36- 37	-.02558	-.00132	1.03632	0.97650	48362.	48605.	-243.	-0.00500
37- 38	0.03740	-.00149	1.03166	0.97557	48099.	48350.	-251.	-0.00520
38- 39	0.04212	-.00080	0.99258	0.97442	46222.	46446.	-224.	-0.00482
39- 40	0.01413	-.00111	0.96597	0.97318	44926.	45141.	-215.	-0.00477
40- 41	0.04378	-.00087	0.93933	0.97191	43630.	43839.	-209.	-0.00477
41- 42	0.00465	-.00159	0.91798	0.97047	42575.	42791.	-216.	-0.00505
42- 43	0.01204	-.00127	0.91165	0.96890	42213.	42424.	-211.	-0.00497
43- 44	-.05336	-.00125	0.93186	0.96724	43074.	43301.	-227.	-0.00523
44- 45	-.00803	-.00128	0.96212	0.96531	44384.	44603.	-219.	-0.00490

45- 46	-.02904	-.00130	0.98138	0.96335	45181.	45421.	-240.	-0.00528
46- 47	0.01687	-.00112	0.98856	0.96149	45424.	45641.	-217.	-0.00475
47- 48	-.04167	-.00069	1.00180	0.95943	45934.	46159.	-225.	-0.00488
48- 49	0.01543	-.00064	1.01571	0.95705	46456.	46673.	-217.	-0.00465
49- 50	-.03263	-.00057	1.02511	0.95455	46763.	46986.	-223.	-0.00474
50- 51	-.03572	-.00066	1.06140	0.95165	48272.	48497.	-225.	-0.00464
51- 52	-.02136	-.00062	1.09283	0.94856	49540.	49764.	-224.	-0.00451
52- 53	-.01986	-.00032	1.11611	0.94525	50419.	50647.	-228.	-0.00451
53- 54	-.02186	-.00050	1.14011	0.94149	51298.	51520.	-222.	-0.00431
54- 55	-.07211	-.00030	1.19543	0.93762	53566.	53830.	-264.	-0.00491
55- 56	-.06339	-.00030	1.27961	0.93340	57080.	57347.	-267.	-0.00466
56- 57	0.19603	-.00041	1.19792	0.92865	53164.	53648.	-484.	-0.00902
57- 58	-.00940	-.00054	1.09171	0.92364	48189.	48399.	-210.	-0.00434
58- 59	-.01965	-.00027	1.10813	0.91846	48639.	48864.	-225.	-0.00460
59- 60	0.01060	0.00008	1.11325	0.91279	48563.	48760.	-197.	-0.00405
60- 61	-.00569	-.00017	1.11057	0.90657	48115.	48303.	-188.	-0.00388
61- 62	-.04069	-.00026	1.13687	0.89955	48874.	49085.	-211.	-0.00430
62- 63	0.00101	-.00006	1.15984	0.89176	49429.	49642.	-213.	-0.00429
63- 64	-.01428	0.00018	1.16750	0.88321	49279.	49495.	-216.	-0.00437
64- 65	0.01673	0.00027	1.16581	0.87376	48681.	48899.	-218.	-0.00446
65- 66	0.00347	-.00019	1.15405	0.86378	47640.	47828.	-188.	-0.00394
66- 67	-.02183	-.00006	1.16484	0.85347	47511.	47724.	-213.	-0.00447
67- 68	0.02133	-.00015	1.16526	0.84190	46884.	47112.	-228.	-0.00484
68- 69	0.02330	-.00029	1.13979	0.82879	45145.	45362.	-217.	-0.00479
69- 70	0.01915	-.00011	1.11607	0.81424	43429.	43651.	-222.	-0.00508
70- 71	0.02131	0.00010	1.09373	0.79804	41713.	41913.	-200.	-0.00477
71- 72	0.02238	-.00027	1.07020	0.77988	39887.	40074.	-187.	-0.00467
72- 73	0.02711	-.00008	1.04422	0.76032	37943.	38136.	-193.	-0.00507
73- 74	-.01075	-.00014	1.03582	0.73885	36575.	36734.	-159.	-0.00434
74- 75	0.02314	-.00031	1.02966	0.71507	35187.	35348.	-161.	-0.00456
75- 76	0.02154	-.00054	1.00734	0.68960	33198.	33342.	-144.	-0.00432
76- 77	0.06110	0.00010	0.96678	0.66130	30554.	30730.	-176.	-0.00574
77- 78	0.00982	-.00025	0.93317	0.62965	28080.	28200.	-120.	-0.00425
78- 79	0.04472	-.00035	0.90833	0.59613	25878.	26035.	-157.	-0.00604
79- 80	0.01364	-.00025	0.88248	0.56100	23659.	23748.	-89.	-0.00373
80- 81	0.02133	0.00028	0.86717	0.52344	21693.	21801.	-108.	-0.00497
81- 82	0.03234	0.00026	0.84398	0.48341	19498.	19605.	-107.	-0.00547
82- 83	0.02121	-.00006	0.82161	0.44154	17337.	17402.	-65.	-0.00373
83- 84	0.04350	0.0	0.79547	0.39994	15204.	15313.	-109.	-0.00712
84- 85	0.01652	0.00008	0.77193	0.35912	13248.	13318.	-70.	-0.00525
85- 86	0.05489	-.00018	0.74489	0.31749	11302.	11389.	-87.	-0.00761
86- 87	0.03983	0.0	0.71050	0.27480	9331.	9391.	-60.	-0.00642
87- 88	0.02623	0.0	0.68741	0.23415	7692.	7702.	-10.	-0.00128
88- 89	0.00202	-.00016	0.67782	0.19750	6398.	6426.	-28.	-0.00439
89- 90	0.04956	0.0	0.66062	0.16398	5177.	5207.	-30.	-0.00576
90- 91	0.04797	0.00049	0.62902	0.13350	4013.	4045.	-32.	-0.00788
91- 92	0.01031	0.00032	0.61071	0.10572	3086.	3105.	-19.	-0.00624
92- 93	0.08788	-.00043	0.58148	0.08304	2308.	2312.	-4.	-0.00189
93- 94	0.08884	0.0	0.53242	0.06382	1624.	1622.	2.	0.00118
94- 95	0.13046	0.0	0.47713	0.04713	1075.	1090.	-15.	-0.01417
95- 96	-.08201	0.0	0.46571	0.03352	746.	732.	14.	0.01920
96- 97	-.00734	0.0	0.48699	0.02371	552.	545.	7.	0.01262
97- 98	0.13005	0.0	0.45801	0.01634	358.	347.	11.	0.03047
98- 99	0.07197	0.0	0.41400	0.01083	214.	209.	5.	0.02547
99-100	0.20822	0.0	0.35988	0.00707	122.	121.	1.	0.00513

single years of age above 100 are listed, N(100) was approximated by

$$N(100) = D(100\text{-}101)e^{r(100^+,\ 1976)/2}$$
$$+$$
$$D(101\text{-}102)e^{r(100^+,\ 1976)+r(100^+,\ 1975)/2}$$
$$+$$
$$D(102\text{-}103)e^{r(100^+,\ 1976)+r(100^+,\ 1975)+r(100^+,\ 1974)/2}$$
$$+$$
$$D(103^+)e^{r(100^+,\ 1976)+r(100^+,\ 1975)+r(100^+,\ 1974)}$$

Results are shown in Table 2.

3) The single-year age distribution of the mean population of Swedish females in 1973-1977 is calculated from the number of female births in 1973-1977, the single-year female life table for the period, the average rate of increase in 1973-1977 of females in each single-year age interval, and the rate of net out-migration at each age. The mean population at each age is one-fifth the number of person-years lived in each single-year age interval during the five-year time period. The growth rate and the net out-migration rate are the increase in the number of persons and the number of net out-migrants, divided by the number of person-years lived during the five years. With rates thus defined, the calculations are based on the same equations as in (1) above that were used for estimating the age distribution of Swedish females in 1976. Results are shown in Table 3.

4) The five-year proportionate age distribution of the mean population of Swedish females in 1976 is calculated from the ${}_5L_x$ function of the Swedish female life table for 1976, and the growth rate in 1976 of the mean population in five-year age intervals. The equation involved is:

$${}_5N_a = {}_5N_o\ e^{-\int_o^a {}_5r_x dx}\ {}_5L_x/{}_5L_o.$$

In this set of calculations, ${}_5r_x$ is taken at five-year intervals, i.e., for x=0, 5, 10, etc., and the integral $\int_o^a {}_5r_x dx$ is approximated by a trapezoid. Results are shown in Table 4.

5) The five-year proportionate age distribution is calculated as in (4), except that ${}_5r_x$ was taken at one-year intervals, i.e., ${}_5r_0$, ${}_5r_1$, ${}_5r_2$, etc., in evaluating the integral $\int_o^a {}_5r_x dx$. Results are shown in Table 5.

The most striking feature of the calculations is the extremely close fit of the calculated data to the accurate Swedish population statistics. In Table 1 the difference between the calculated and recorded populations does not exceed one percent until age 94, and in Table 2 until age 85, with the exception of age 17. Incredibly enough, the relatively large discrepancy at age 17 is the result of an error in the Swedish yearbook for 1976. The mean population is listed in Table 4:15, which presents the life table for 1976. It is readily verified that the mean population at each age as listed in this table is simply the arithmetic average of the year-end populations for 1975 and 1976 listed elsewhere; the mean population at age 17-18 in 1976 calculated in this way is 51,644 instead of the listed 52,144. This is an error of 500 persons, which doubtless occurred as the result of a punching mistake of one digit in the thousands column for 17 year olds in year-end 1975 or 1976 when the mean population was calculated. The precision of these calculations thus proves to be sufficient to detect an isolated one percent error in the listing of the single-year mid-year population of Swedish females.

A more significant result of the precision of the calculation is the close agreement of the calculated populations from 90 to 100 with the official figures. If the official Swedish life table is employed in calculating Table 1, the agreement is much poorer. The published life table for 1976 (and other years) is based on Wittstein's formula ($q_x = a^{-(M-x)^n}$) above age 91 rather than directly on recorded numbers of deaths and persons. The difference between the official table of 1976 and the table we constructed, and its effect on the estimated population from age 92-93 to 99-100, are as follows:

	${}_1L_x/\ell_o$		Proportionate error in estimated population with:	
x	Official	Calculated	Official life table	Calculated
92	.08313	.08304	.003	.002
93	.06314	.06382	-.012	-.001
94	.04651	.04713	.007	.014
95	.03308	.03352	-.032	-.019
96	.02260	.02371	-.059	-.013
97	.01474	.01634	-.125	-.030
98	.00910	.01083	-.181	-.025
99	.00526	.00707	-.260	-.005

We calculated a life table above age 91 by accepting the official ℓ_{91}, and from x = 91 to 99, estimating ℓ_{x+1}/ℓ_x as $e^{-{}_1M_x}$, and ${}_1L_x$ as $\sqrt{\ell_x \cdot \ell_{x+1}}$. The official life table produces estimates in which errors increase rapidly above age 95; evidently the unadjusted death rates are a more realistic basis for a life table than those calculated by the Wittstein formula.

Table 2: Number of females in Sweden in 1976, by single years of age, calculated from $N(a) = \int_a^w (D(x) + E(x)) e^{\int_a^x r(y)dy} dx$, compared with recorded mean population.

Age	Rate of Increase r(x)	Deaths D(x)	Net Emigrants E(x)	Estimated Number at Recording Age at N(a)	Estimated Number a to a+1 $_1N_a$	Recorded Mean Population	Estimated -Recorded	Proportionate Error
0- 1	-.04894	358.	-459.	47841.	49076.	49054.	22.	0.00046
1- 2	-.05537	25.	-574.	50344.	52031.	52371.	-340.	-.00650
2- 3	0.01052	15.	-348.	53775.	53659.	53901.	-242.	-.00450
3- 4	-.01486	9.	-227.	53543.	54051.	54255.	-204.	-.00376
4- 5	-.00692	17.	-193.	54564.	54841.	55033.	-192.	-.00349
5- 6	0.03749	18.	-229.	55120.	54201.	54420.	-219.	-.00402
6- 7	0.02390	10.	-179.	53299.	52750.	52940.	-190.	-.00359
7- 8	-.05674	13.	-96.	52207.	53751.	53928.	-177.	-.00329
8- 9	-.05990	8.	-126.	55340.	57082.	57296.	-214.	-.00374
9- 10	-.00924	17.	-120.	58878.	59202.	59401.	-199.	-.00334
10- 11	-.00244	9.	-139.	59528.	59666.	59880.	-214.	-.00359
11- 12	-.00301	15.	-97.	59804.	59935.	60127.	-192.	-.00319
12- 13	0.08868	11.	-104.	60066.	57508.	57750.	-242.	-.00420
13- 14	0.04765	6.	-85.	55058.	53801.	53987.	-186.	-.00344
14- 15	0.02742	17.	-77.	52573.	51887.	52049.	-162.	-.00311
15- 16	0.01900	20.	-107.	51210.	50770.	50943.	-173.	-.00340
16- 17	-.01336	20.	-195.	50333.	50757.	50974.	-217.	-.00425
17- 18	-.00310	21.	-260.	51186.	51384.	52144.	-760.	-.01457
18- 19	-.02129	30.	-346.	51584.	52294.	52604.	-310.	-.00590
19- 20	0.00710	26.	-554.	53014.	53089.	53499.	-410.	-.00767
20- 21	0.00026	20.	-577.	53164.	53435.	53861.	-426.	-.00791
21- 22	0.02117	33.	-535.	53708.	53392.	53802.	-410.	-.00762
22- 23	-.03333	35.	-531.	53079.	54219.	54640.	-421.	-.00771
23- 24	-.00080	28.	-493.	55383.	55637.	56035.	-398.	-.00711
24- 25	-.00135	23.	-378.	55892.	56107.	56452.	-345.	-.00611
25- 26	-.04017	27.	-317.	56323.	57610.	57943.	-333.	-.00575
26- 27	-.05212	32.	-271.	58927.	60602.	60927.	-325.	-.00533
27- 28	-.04056	38.	-197.	62325.	63681.	63666.	15.	0.00024
28- 29	-.00782	33.	-128.	65067.	65370.	65598.	-228.	-.00348
29- 30	-.01659	43.	-129.	65674.	66264.	66494.	-231.	-.00347
30- 31	0.00896	44.	-120.	66859.	66598.	66816.	-219.	-.00327
31- 32	0.01496	42.	-110.	66338.	65877.	66090.	-213.	-.00322
32- 33	0.05771	44.	-98.	65420.	63586.	63805.	-219.	-.00343
33- 34	0.09237	50.	-73.	61804.	59026.	59249.	-223.	-.00376
34- 35	0.11138	50.	-57.	56373.	53323.	53547.	-224.	-.00419
35- 36	0.05332	49.	-71.	50438.	49122.	49278.	-156.	-.00317
36- 37	-.02558	37.	-64.	47840.	48469.	48605.	-136.	-.00280
37- 38	0.03740	55.	-72.	49107.	48205.	48350.	-145.	-.00299
38- 39	0.04212	57.	-37.	47321.	46325.	46446.	-121.	-.00261
39- 40	0.01413	59.	-50.	45350.	45026.	45141.	-115.	-.00255
40- 41	0.04378	56.	-38.	44704.	43727.	43839.	-112.	-.00255
41- 42	0.00465	72.	-68.	42772.	42670.	42791.	-121.	-.00282
42- 43	0.01204	65.	-54.	42569.	42308.	42424.	-116.	-.00273
43- 44	-.05336	82.	-54.	42049.	43171.	43301.	-130.	-.00299
44- 45	-.00803	93.	-57.	44325.	44485.	44603.	-118.	-.00265

45- 46	-.02904	89.	-59.	44646.	45284.	45421.	-137.	-.00302
46- 47	0.01687	87.	-51.	45931.	45527.	45641.	-114.	-.00250
47- 48	-.04167	109.	-32.	45127.	46038.	46159.	-121.	-.00262
48- 49	0.01543	121.	-30.	46968.	46562.	46673.	-111.	-.00238
49- 50	-.03263	124.	-27.	46159.	46870.	46986.	-116.	-.00247
50- 51	-.03572	167.	-32.	47591.	48381.	48497.	-116.	-.00239
51- 52	-.02136	152.	-31.	49185.	49652.	49764.	-112.	-.00225
52- 53	-.01986	199.	-16.	50124.	50533.	50647.	-114.	-.00226
53- 54	-.02186	207.	-26.	50945.	51414.	51520.	-106.	-.00206
54- 55	-.07211	227.	-16.	51888.	53687.	53830.	-143.	-.00266
55- 56	-.06339	275.	-17.	55549.	57208.	57347.	-139.	-.00242
56- 57	0.19603	290.	-22.	58918.	53282.	53648.	-366.	-.00681
57- 58	-.00940	262.	-26.	48187.	48296.	48399.	-103.	-.00214
58- 59	-.01965	285.	-13.	48405.	48746.	48864.	-118.	-.00241
59- 60	0.01060	319.	4.	49090.	48669.	48760.	-91.	-.00186
60- 61	-.00569	344.	-8.	48251.	48220.	48303.	-83.	-.00171
61- 62	-.04069	413.	-13.	48190.	48980.	49085.	-105.	-.00214
62- 63	0.00101	446.	-3.	49783.	49536.	49642.	-106.	-.00213
63- 64	-.01428	508.	9.	49290.	49384.	49495.	-111.	-.00224
64- 65	0.01673	550.	13.	49479.	48784.	48899.	-115.	-.00235
65- 66	0.00347	560.	-9.	48100.	47740.	47828.	-88.	-.00184
66- 67	-.02183	588.	-3.	47383.	47610.	47724.	-114.	-.00240
67- 68	0.02133	705.	-7.	47837.	46980.	47112.	-132.	-.00281
68- 69	0.02330	745.	-13.	46137.	45235.	45362.	-127.	-.00280
69- 70	0.01915	829.	-5.	44351.	43514.	43651.	-137.	-.00313
70- 71	0.02131	888.	4.	42694.	41793.	41913.	-120.	-.00287
71- 72	0.02238	996.	-11.	40911.	39960.	40074.	-114.	-.00284
72- 73	0.02711	989.	-3.	39031.	38010.	38136.	-126.	-.00332
73- 74	-.01075	1154.	-5.	37015.	36635.	36734.	-99.	-.00270
74- 75	0.02314	1203.	-11.	36259.	35241.	35348.	-107.	-.00302
75- 76	0.02154	1286.	-18.	34252.	33245.	33342.	-97.	-.00292
76- 77	0.06110	1394.	3.	32268.	30590.	30730.	-140.	-.00455
77- 78	0.00982	1490.	-7.	29000.	28107.	28200.	-93.	-.00331
78- 79	0.04472	1473.	-9.	27241.	25896.	26035.	-139.	-.00533
79- 80	0.01364	1545.	-6.	24618.	23669.	23748.	-79.	-.00334
80- 81	0.02133	1606.	6.	22756.	21693.	21801.	-108.	-.00493
81- 82	0.03234	1678.	5.	20681.	19489.	19605.	-116.	-.00590
82- 83	0.02121	1665.	-1.	18367.	17321.	17402.	-81.	-.00466
83- 84	0.04350	1562.	0.	16335.	15182.	15313.	-131.	-.00854
84- 85	0.01652	1505.	1.	14111.	13221.	13318.	-97.	-.00731
85- 86	0.05489	1520.	-2.	12386.	11266.	11389.	-123.	-.01076
86- 87	0.03983	1458.	0.	10248.	9298.	9391.	-103.	-.01095
87- 88	0.02623	1265.	0.	8418.	7650.	7702.	-52.	-.00674
88- 89	0.00202	1128.	-1.	6952.	6356.	6426.	-70.	-.01082
89- 90	0.04956	1019.	0.	5812.	5135.	5207.	-72.	-.01381
90- 91	0.04797	864.	2.	4537.	3973.	4045.	-72.	-.01781
91- 92	0.01031	757.	1.	3479.	3059.	3105.	-46.	-.01490
92- 93	0.08788	553.	-1.	2689.	2281.	2312.	-31.	-.01341
93- 94	0.08884	466.	0.	1935.	1601.	1622.	-21.	-.01306
94- 95	0.13046	348.	0.	1325.	1053.	1090.	-37.	-.03432
95- 96	-.08201	265.	0.	836.	727.	732.	-5.	-.00678
96- 97	-.00734	180.	0.	632.	537.	545.	-8.	-.01517
97- 98	0.13005	144.	0.	456.	348.	347.	1.	0.00236
98- 99	0.07197	85.	0.	265.	209.	209.	0.	0.00101
99-100	0.20822	54.	0.	165.	119.	121.	-2.	-.01986

Table 3: Number of person-years lived by females in Sweden, 1973-1977, calculated from $N(a) = Be^{-\int_0^a (r(x) + e(x))dx} p(a)$, compared with recorded mean population, by single years of age.

Age	Rate of Increase r(x)	Deaths D(x)	Net Emigrants E(x)	Estimated Number at Recording Age at N(a)	Estimated Number a to a+1 $_1N_a$	Recorded Mean Population	Estimated -Recorded	Proportionate Error
0- 1	-.03111	-.00594	1.01870	0.99289	254131.	254657.	-526.	-0.00206
1- 2	-.02496	-.00886	1.05544	0.99181	263012.	264099.	-1087.	-0.00411
2- 3	-.00762	-.00393	1.07966	0.99138	268931.	269526.	-595.	-0.00221
3- 4	0.00771	-.00238	1.08303	0.99112	269697.	270194.	-498.	-0.00184
4- 5	-.00552	-.00162	1.08401	0.99086	269870.	270326.	-456.	-0.00169
5- 6	-.01513	-.00151	1.09698	0.99057	273019.	273542.	-523.	-0.00191
6- 7	-.01510	-.00082	1.11498	0.99028	277420.	277880.	-460.	-0.00165
7- 8	-.02206	-.00037	1.13657	0.99003	282719.	283167.	-448.	-0.00158
8- 9	-.02616	-.00081	1.16499	0.98982	289727.	290294.	-567.	-0.00195
9- 10	0.00170	-.00097	1.18037	0.98962	293493.	294114.	-621.	-0.00211
10- 11	0.02256	-.00102	1.16730	0.98942	290183.	290835.	-652.	-0.00224
11- 12	0.03034	-.00084	1.13789	0.98923	282819.	283469.	-650.	-0.00229
12- 13	0.03553	-.00136	1.10223	0.98902	273899.	274616.	-717.	-0.00261
13- 14	0.03487	-.00121	1.06547	0.98879	264702.	265370.	-668.	-0.00252
14- 15	0.01553	-.00117	1.04019	0.98853	258354.	258967.	-613.	-0.00237
15- 16	0.00076	-.00145	1.03310	0.98823	256515.	257133.	-618.	-0.00240
16- 17	-.00322	-.00298	1.03667	0.98785	257303.	258165.	-863.	-0.00334
17- 18	-.00760	-.00431	1.04611	0.98746	259541.	261125.	-1584.	-0.00607
18- 19	-.00130	-.00564	1.05601	0.98703	261885.	263170.	-1285.	-0.00488
19- 20	-.00787	-.00805	1.06816	0.98657	264774.	266454.	-1680.	-0.00631
20- 21	-.00433	-.00837	1.08356	0.98617	268480.	270271.	-1791.	-0.00663
21- 22	-.00557	-.00762	1.09767	0.98575	271864.	273638.	-1774.	-0.00648
22- 23	-.01421	-.00644	1.11640	0.98532	276380.	278057.	-1677.	-0.00603
23- 24	-.02875	-.00487	1.14711	0.98488	283855.	285402.	-1547.	-0.00542
24- 25	-.02995	-.00432	1.18672	0.98444	293528.	295106.	-1578.	-0.00535
25- 26	-.03019	-.00282	1.22732	0.98402	303439.	304849.	-1410.	-0.00463
26- 27	-.03240	-.00229	1.26957	0.98355	313738.	315144.	-1406.	-0.00446
27- 28	-.02202	-.00109	1.30681	0.98306	322775.	323717.	-942.	-0.00291
28- 29	-.00892	-.00076	1.32841	0.98255	327942.	329131.	-1189.	-0.00361
29- 30	0.01017	-.00093	1.32871	0.98203	327843.	329041.	-1199.	-0.00364
30- 31	0.02920	-.00060	1.30381	0.98146	321514.	322631.	-1117.	-0.00346
31- 32	0.05316	-.00071	1.25203	0.98084	308547.	309686.	-1139.	-0.00368
32- 33	0.06344	-.00070	1.18196	0.98018	291084.	292188.	-1104.	-0.00378
33- 34	0.05914	-.00031	1.11225	0.97947	273719.	274647.	-928.	-0.00338
34- 35	0.05604	-.00034	1.05035	0.97869	258280.	259116.	-836.	-0.00323
35- 36	0.04443	-.00059	0.99935	0.97790	245542.	246343.	-801.	-0.00325
36- 37	0.02382	-.00051	0.96636	0.97708	237236.	237944.	-708.	-0.00297
37- 38	0.02142	-.00073	0.94534	0.97613	231850.	232568.	-718.	-0.00309
38- 39	0.02856	-.00060	0.92262	0.97502	226020.	226712.	-692.	-0.00305
39- 40	0.02316	-.00072	0.89966	0.97384	220129.	220791.	-663.	-0.00300
40- 41	0.00379	-.00060	0.88820	0.97262	217053.	217669.	-616.	-0.00283
41- 42	-.00120	-.00073	0.88764	0.97127	216616.	217262.	-646.	-0.00297
42- 43	-.01550	-.00040	0.89559	0.96980	218224.	218835.	-611.	-0.00279
43- 44	-.01279	-.00061	0.90880	0.96816	221069.	221721.	-652.	-0.00294
44- 45	-.02321	-.00056	0.92585	0.96630	224783.	225464.	-681.	-0.00302

45- 46	-.00970	-.00085	0.94188	0.96438	228220.	228941.	-721.	-0.00315
46- 47	-.01467	-.00054	0.95409	0.96247	230721.	231413.	-693.	-0.00299
47- 48	-.01627	-.00038	0.96941	0.96025	233884.	234574.	-690.	-0.00294
48- 49	-.02346	-.00046	0.98928	0.95769	238041.	238770.	-729.	-0.00305
49- 50	-.01934	-.00042	1.01112	0.95497	242606.	243365.	-759.	-0.00312
50- 51	-.02642	-.00039	1.03494	0.95198	247544.	248323.	-779.	-0.00314
51- 52	-.03483	-.00025	1.06746	0.94887	254489.	255273.	-784.	-0.00307
52- 53	-.04101	-.00014	1.10893	0.94551	263440.	264318.	-878.	-0.00332
53- 54	0.00324	-.00024	1.13029	0.94173	267442.	268239.	-797.	-0.00297
54- 55	0.00538	-.00024	1.12570	0.93771	265216.	266042.	-826.	-0.00310
55- 56	0.00572	-.00020	1.11971	0.93342	262600.	263418.	-818.	-0.00310
56- 57	0.02285	-.00021	1.10406	0.92871	257621.	258419.	-798.	-0.00309
57- 58	0.03706	-.00034	1.07177	0.92361	248716.	249536.	-821.	-0.00329
58- 59	-.01297	-.00015	1.05919	0.91822	244362.	245120.	-758.	-0.00309
59- 60	-.01083	-.00011	1.07201	0.91248	245771.	246508.	-737.	-0.00299
60- 61	-.00981	-.00017	1.08328	0.90621	246650.	247408.	-758.	-0.00306
61- 62	-.00792	-.00008	1.09306	0.89926	246968.	247733.	-765.	-0.00309
62- 63	-.00639	-.00005	1.10097	0.89180	246692.	247506.	-814.	-0.00329
63- 64	-.00303	0.00004	1.10618	0.88368	245601.	246362.	-761.	-0.00309
64- 65	0.00084	-.00000	1.10737	0.87452	243319.	244119.	-800.	-0.00328
65- 66	0.00825	-.00019	1.10246	0.86449	239459.	240250.	-791.	-0.00329
66- 67	0.00847	-.00011	1.09345	0.85354	234495.	235284.	-790.	-0.00336
67- 68	0.01167	-.00012	1.08262	0.84140	228868.	229733.	-865.	-0.00376
68- 69	0.02155	-.00017	1.06493	0.82816	221588.	222421.	-833.	-0.00374
69- 70	0.02283	-.00020	1.04175	0.81351	212930.	213751.	-822.	-0.00384
70- 71	0.01682	-.00012	1.02146	0.79730	204623.	205446.	-823.	-0.00401
71- 72	0.01662	-.00026	1.00472	0.77943	196757.	197592.	-835.	-0.00423
72- 73	0.01663	-.00020	0.98838	0.75961	188637.	189418.	-781.	-0.00412
73- 74	0.02342	-.00018	0.96898	0.73803	179680.	180481.	-801.	-0.00444
74- 75	0.01991	-.00026	0.94842	0.71445	170249.	170916.	-667.	-0.00390
75- 76	0.03141	-.00031	0.92466	0.68838	159927.	160701.	-774.	-0.00481
76- 77	0.03065	-.00017	0.89662	0.65965	148606.	149379.	-773.	-0.00518
77- 78	0.03187	-.00014	0.86916	0.62856	137264.	137930.	-666.	-0.00483
78- 79	0.02620	-.00022	0.84444	0.59544	126334.	126944.	-610.	-0.00481
79- 80	0.02936	-.00020	0.82148	0.56025	115634.	116212.	-578.	-0.00497
80- 81	0.02816	-.00021	0.79835	0.52347	105002.	105647.	-645.	-0.00611
81- 82	0.02917	-.00002	0.77588	0.48493	94532.	95075.	-543.	-0.00571
82- 83	0.03709	-.00018	0.75067	0.44462	83860.	84412.	-552.	-0.00654
83- 84	0.03815	-.00004	0.72303	0.40364	73327.	73840.	-513.	-0.00695
84- 85	0.03913	0.0	0.69564	0.36261	63378.	63813.	-435.	-0.00681
85- 86	0.03335	-.00026	0.67097	0.32118	54145.	54539.	-394.	-0.00722
86- 87	0.04193	-.00017	0.64632	0.27992	45456.	45810.	-354.	-0.00773
87- 88	0.04079	-.00008	0.62021	0.24060	37493.	37684.	-191.	-0.00508
88- 89	0.03408	-.00010	0.59748	0.20371	30580.	30810.	-230.	-0.00746
89- 90	0.04932	-.00020	0.57316	0.17002	24484.	24654.	-170.	-0.00690
90- 91	0.06882	0.00005	0.54033	0.13957	18948.	19152.	-204.	-0.01064
91- 92	0.06721	-.00007	0.50480	0.11213	14222.	14358.	-136.	-0.00949
92- 93	0.06554	-.00009	0.47242	0.08847	10501.	10574.	-73.	-0.00688
93- 94	0.06597	0.0	0.44238	0.06767	7521.	7625.	-104.	-0.01359
94- 95	0.07715	0.0	0.41183	0.05007	5181.	5379.	-198.	-0.03689
95- 96	0.07712	-.00084	0.38142	0.03628	3476.	3566.	-90.	-0.02514
96- 97	0.04266	0.0	0.35940	0.02541	2295.	2438.	-143.	-0.05871
97- 98	0.09190	0.0	0.33602	0.01694	1430.	1567.	-137.	-0.08718
98- 99	0.09691	-.00206	0.30606	0.01072	824.	970.	-146.	-0.15025
99-100	0.07241	0.0	0.28151	0.00622	440.	580.	-140.	-0.24205

Table 4: Proportionate distribution of female population in Sweden in 1976, by five-year age intervals, calculated from

$${}_5C_a = {}_5C_o e^{-\int_o^a ({}_5r_x + {}_5e_x)dx} \; {}_5L_a / {}_5L_o$$, compared with recorded distribution.

Age	Growth Rate (x Five) $5.\ {}_5r_x$	Rate of Out-Migration (x Five) $5\,{}_5e_x$	${}_5L_a/{}_5L_0$	${}_5C_a/{}_5C_0$	Estimated ${}_5C_a$	Recorded ${}_5C_a$	Estimated -Recorded	Proportion of Recorded
0- 4	-.11185	1.00000	1.00000	1.00000	0.06176	0.06399	-0.00222	-0.03600
5- 9	-.06716	1.00677	0.99854	1.11829	0.06907	0.06722	0.00185	0.02677
10- 14	0.15488	0.99417	0.99749	1.08119	0.06678	0.06863	-0.00185	-0.02766
15- 19	-.01180	1.00148	0.99587	1.02368	0.06323	0.06279	0.00044	0.00689
20- 24	-.01436	1.02013	0.99340	1.07352	0.06630	0.06645	-0.00014	-0.00217
25- 29	-.15421	1.01423	0.99083	1.20175	0.07422	0.07615	-0.00193	-0.02600
30- 34	0.26978	0.99497	0.98767	1.14428	0.07067	0.07484	-0.00417	-0.05898
35- 39	0.12165	0.98983	0.98319	0.94298	0.05824	0.05751	0.00073	0.01259
40- 44	-.00090	0.98926	0.97649	0.88718	0.05480	0.05246	0.00233	0.04256
45- 49	-.07114	0.98833	0.96682	0.91544	0.05654	0.05583	0.00071	0.01257
50- 54	-.17321	0.98642	0.95245	1.02243	0.06315	0.06148	0.00167	0.02638
55- 59	0.11577	0.98501	0.93075	1.03021	0.06363	0.06215	0.00148	0.02324
60- 64	-.04301	0.98386	0.89807	0.95924	0.05925	0.05935	-0.00010	-0.00170
65- 69	0.04364	0.98354	0.84714	0.90493	0.05589	0.05602	-0.00013	-0.00235
70- 74	0.08447	0.98385	0.76448	0.76653	0.04734	0.04648	0.00087	0.01829
75- 79	0.15347	0.98410	0.63254	0.56365	0.03481	0.03435	0.00046	0.01328
80- 84	0.13461	0.98346	0.44501	0.34347	0.02121	0.02114	0.00007	0.00329
85- 89	0.18350	0.98300	0.23948	0.15763	0.00974	0.00970	0.00004	0.00368
90- 94	0.29375	0.98291	0.08733	0.04527	0.00280	0.00294	-0.00015	-0.05278
95- 99	0.05377	0.98272	0.01844	0.00803	0.00050	0.00047	0.00002	0.04817
100 +	0.41825	0.98313	0.00066	0.00112	0.00007	0.00004	0.00003	0.45726

$\sum_o^{100} {}_5C_a/{}_5C_o = 16.191$; ${}_5C_o = 1/16.191$

(In this table $\int_o^a {}_5r_x d_x$ is estimated from values of ${}_5r_x$ at x=0, 5, 10, etc.)

Table 5: Proportionate distribution of female population in Sweden in 1976, by five-year age intervals, calculated from

$$ {}_5C_a = {}_5C_o e^{-\int_o^a ({}_5r_x + {}_5e_x)dx} \, {}_5L_a / {}_5L_o, $$ compared with recorded distribution.

Age	$e^{-\int_{o}^{a} {}_5r_x d_x}$	$e^{-\int_{o}^{a} {}_5e_x d_x}$	${}_5L_a / {}_5L_o$	${}_5C_a / {}_5C_o$	Estimated ${}_5C_a$	Recorded ${}_5C_a$	Estimated -Recorded	Proportion of Recorded
0- 4	1.00000	1.00000	1.00000	1.00000	0.06400	0.06398	0.00002	0.00027
5- 9	0.97438	1.00433	0.99854	1.05110	0.06727	0.06722	0.00005	0.00081
10- 14	0.95636	0.99342	0.99749	1.07305	0.06868	0.06862	0.00006	0.00080
15- 19	0.85279	0.99712	0.99587	0.97991	0.06272	0.06291	-0.00019	-0.00307
20- 24	0.95904	1.02596	0.99340	1.03454	0.06621	0.06644	-0.00023	-0.00352
25- 29	1.05601	1.01421	0.99083	1.18877	0.07608	0.07608	0.00001	0.00007
30- 34	0.92354	0.99352	0.98767	1.16955	0.07485	0.07484	0.00001	0.00018
35- 39	0.72551	0.98964	0.98319	0.89916	0.05755	0.05751	0.00004	0.00073
40- 44	0.86355	0.98891	0.97649	0.82034	0.05250	0.05246	0.00004	0.00080
45- 49	1.01062	0.98881	0.96682	0.87307	0.05588	0.05583	0.00005	0.00091
50- 54	1.05410	0.98622	0.95245	0.96178	0.06156	0.06148	0.00007	0.00122
55- 59	0.97650	0.98501	0.93075	0.97243	0.06224	0.06215	0.00009	0.00143
60- 64	0.93520	0.98393	0.89807	0.92872	0.05944	0.05934	0.00010	0.00160
65- 69	0.94584	0.98326	0.84714	0.87637	0.05609	0.05602	0.00007	0.00123
70- 74	0.86828	0.98375	0.76448	0.72664	0.04651	0.04648	0.00003	0.00066
75- 79	0.84254	0.98427	0.63254	0.53632	0.03433	0.03435	-0.00002	-0.00070
80- 84	0.82487	0.98339	0.44501	0.32922	0.02107	0.02114	-0.00007	-0.00342
85- 89	0.80220	0.98315	0.23948	0.15030	0.00962	0.00970	-0.00008	-0.00834
90- 94	0.78191	0.98281	0.08733	0.04531	0.00290	0.00294	-0.00004	-0.01513
95- 99	0.72685	0.98320	0.01844	0.00735	0.00047	0.00047	-0.00000	-0.00381
100-104	0.53501	0.98313	0.00066	0.00074	0.00005	0.00004	0.00001	0.20163

$\sum_{o}^{100} {}_5C_a / {}_5C_o = 15.62; \quad {}_5C_o = 1/15.62$

(In this table $\int_{o}^{a} {}_5r_x d_x$ is estimated from values of ${}_5r_x$ at x=0, 1, 2, 3, 4, 5, etc.)

Note that the proportionate age distribution is even more accurately estimated than the absolute numbers. The estimated population is consistently smaller than the recorded by about 0.005 times the recorded number in Table A-1, and about .0035 times the recorded number in Table A-2.

The estimation of the single-year age distribution of person-years lived in 1973-1977 is equally precise, with a typical proportionate underestimate of about 0.003 times the recorded number, until ages above 90.

Calculation of the age distribution by five-year age intervals produces an estimate of substantially less precision than the single-year estimates, when growth rates of five-year age groups are utilized only at intervals of five years. (See Table 4, where the error reaches almost six percent of the true proportion.) The reason for this greater error is that the proper identity is

$${}_5N_a = {}_5N_o e^{-\int_o^a ({}_5r_x + {}_5e_x)dx} \, {}_5L_a/{}_5L_o,$$

so that the precise calculation calls for the evaluation of the integral of a function $({}_5r_x + {}_5e_x)$ that is a continuous function of age. The integral of ${}_5r_x$ from o to <u>a</u> is really something like

$$\frac{{}_5r_o}{20} + \frac{{}_5r_{o.1} + \cdots {}_5r_{a-o.1}}{10} + \frac{{}_5r_a}{20} \; .$$

In constructing Table 4, $\int_o^a {}_5r_x dx$ was approximated by a trapezoidal formula using values of ${}_5r_o$, ${}_5r_5$, etc., as $2.5({}_5r_o)+5({}_5r_5+ \ldots {}_5r_{a-5})+2.5({}_5r_a)$, analogous to estimating the integral of any continuous function by five-year wide trapezoids. Since, in Sweden, the irregular age distribution caused by past variations in fertility causes an erratic sequence of age-specific growth rates, the trapezoidal approximation at five-year intervals is not a very close approximation.

In Table 5 the age distribution by five-year intervals has been calculated on the basis of the same equation, but with five-year growth rates (and emigration rates) taken at starting ages only one year apart. In other words, $\int_o^a {}_5r_x dx$ is calculated by a trapezoidal approximation, but with one-year wide trapezoids; namely $\int_o^a {}_5r_x dx \approx {}_5r_o/2 + {}_5r_1 + {}_5r_2 + \ldots {}_5r_{a-1} + {}_5r_a/2$. Note that in Table 5 this calculation has produced an age distribution that fits the recorded distribution with extraordinary precision.

As a last point in this illustrative use of Swedish data, we have calculated the net reproduction rate for each year from 1973 to 1977 from the formula $NRR = \int_\alpha^\beta e^{-\int_o^a r(x)dx} v(a)da$, where v(a) is the proportion of the total number of births occurring to women at age a. The sequence is 0.889, 0.896, 0.849, 0.809, 0.792, compared with the official calculations of 0.896, 0.899, 0.851, 0.806, 0.785 -- an error of less than one percent in every year in calculating the net reproduction rate without explicit use of mortality data, or of the level of fertility.

Applications for Estimation from Limited Data

a) Mortality

The formulation in (3) for a closed population can be used to infer intercensal mortality conditions from two census age distributions. Recognizing that life expectancy at birth is

$$e_o^o = \int_o^a p(a)da,$$

one can simply integrate both sides of equation (3) to estimate e_o^o as

$$e_o^o = \int_o^\infty \frac{N(a)}{N(0)} e^{\int_o^a r(x)dx} da \quad .$$

Generally, estimates of N(0) will be poor. Higher starting points can usually be more accurately estimated by averaging successive segments of the age distribution. For example, life expectancy at age 5 is

$$e_5^o = \int_5^\infty \frac{N(a)}{N(5)} e^{\int_5^a r(x)dx} da \quad .$$

N(5) can be estimated as one-tenth of the total population between ages 0 and 10. Preston and Bennett (1982) have shown that this estimation system gives good results in Sweden, India, and the Republic of Korea. It is always subject to the quality of census data, of course, and seems to work substantially less well in Kenya (Hill, 1981).

Directly inferring mortality from two age distributions means that errors in the latter will often affect the former. Partly for this reason, demographers have developed "model" life tables that impose regularity on the age sequence of p(a)'s and thus help to smooth out distortions in the age distributions. All of the estimation methods that combine model life tables and stable population analysis can be adapted to the more general case. For example, Coale and Demeny (1967) recommend using the cumulative proportion below certain ages, in combination with the stable growth rate, to identify the correct level of mortality within a model life table system. Age 35 is often considered a good choice for estimation purposes. The new formula for the proportion below age 35 is

$$C(35) = \frac{\int_0^{35} e^{-\int_0^a r(x)dx} p(a)da}{\int_0^{\infty} e^{-\int_0^a r(x)dx} p(a)da} .$$

Solving for the current level of mortality thus involves substituting trial values of the p(a) function among candidates drawn from a model life table system until a set is found that equates the right-hand side to the observed value of C(35). Higher levels of life expectancy will produce lower values of C(35), given the observed set of r(x)'s.

An alternative procedure is to use Brass's (1975) one-parameter transformation of age-specific death rates. Assume that

$$\frac{q(a)}{p(a)} = \kappa \frac{q_s(a)}{p_s(a)} ,$$

where $q(a) = 1 - p(a)$

$q_s(a)$, $p_s(a)$ = q(a) and p(a) functions in the model life table adopted as a standard

κ = parameter representing level of mortality in the population.

After substituting into (5) and simplification, we find that

$$\frac{e^{-\int_0^a r(x)dx}}{c(a)} = \frac{1}{b} + \frac{\kappa}{b} \cdot \frac{q_s(a)}{p_s(a)} .$$

This is now a simple linear equation whose intercept is the reciprocal of the birth rate and whose slope is the product of the intercept and κ. Preston (1982) applies this procedure in several countries with promising results.

By generalizing stable population relations the new equations seem certain to displace the estimation procedures based upon quasi-stable methods (e.g., Coale and Demeny, 1967). These involved simulations of the effect of mortality change on population age structures and growth rates. The analyst then attempts to locate the simulation appropriate to his situation by referring to the growth history of the population under study. But we have seen that all of the features of that history that are pertinent to demographic estimation are contained in the series of contemporaneous age-specific growth rates.

Another data situation pertains when registered deaths are available by age. If death registration is complete, of course, no indirect estimation of mortality is required. But often the level of completeness is unknown. As Bennett and Horiuchi (1981) have shown, it is possible to use the system to estimate the completeness of registration. As demonstrated above,

$$d(a) = \frac{D(a)e^{\int_0^a r(x)dx}}{\int_0^\infty D(a)e^{\int_0^a r(x)dx} da} .$$

D(a) is simply observed deaths at age a, and $d(a) = p(a)\mu(a)$ is deaths in the underlying life table at age a corresponding to current mortality conditions (with radix of one). Integrated from 0 to ∞, the d(a) function must equal unity. Thus

$$\frac{\int_0^\infty D(a)\, e^{\int_0^a r(x)dx} da}{N(0)} = 1 . \qquad (16)$$

However, the left-hand side of equation (16) will equal unity only if deaths are completely registered. If they are registered with completeness C at all ages, then the value of the left-hand side will equal C. Therefore, its value provides a direct estimate of registration completeness.[3] Equation (16) can be implemented from any starting age and need not begin at zero, since the probability of dying above age y (the arbitrary starting age) for someone who survived to that age is always unity.

Estimates are less vulnerable to error in the N(0) or N(y) series if the registered deaths are compared with the total population above 0 or y. This improvement can be introduced by integrating over age for a second time. In this case the formula for C starting from arbitrary age y is

$$\frac{\int_y^\infty \int_y^\infty D(a)\, e^{\int_y^a r(x)dx} da\, da}{\int_y^\infty N(a)da} = C . \qquad (17)$$

Bennett and Horiuchi (1981) have shown that equation (17) gives very good results in Sweden and the Republic of Korea. Note that, after solving for C in the more robust formula (17), one can then take the estimated value, insert it into (16) to correct the D(a) series, and use (16) to estimate the "true" number of births, N(0). Thus, registered deaths by age and age-specific growth rates are sufficient to estimate the birth rate. Using them in this fashion requires the assumption that C is invariant to age, which may be untenable for infancy.

The system in (17) can give different and hence inconsistent estimates of C for different starting ages. A fitting procedure is available to produce a synthetic estimate. If deaths are registered with completeness C relative to the completeness of population enumeration, then in a life table produced from the data,

$$p_T(a) = p_R(a)^{1/C}$$

where $p_R(a)$ is the probability of surviving to age a in the life table produced by the data and $p_T(a)$ is the true probability under prevailing mortality conditions. Substituting this expression into equation (5), taking logs and rearranging, we have

$$\ell n c(a) - \int_o^a r(x)dx = \ell n b + \frac{1}{C} \ell n \; p_R(a)$$

$$= \ell n b - \frac{1}{C} \int_o^a \mu_R(x)dx \; .$$

This is again a simple linear equation whose intercept is the log of the birth rate and whose slope is the reciprocal of registration completeness. The independent variable is simply the sum of recorded age-specific growth rates up to age a.

While this system of equations is useful for estimating registration completeness, it can also be used to infer mortality (and fertility) conditions directly from two sets of deaths by age. If we are prepared to assume that mortality is constant over the period of observation, then

$$r(x, \; t \text{ to } t+n) = \frac{\ell n \left[\frac{D(x, \; t+n)}{D(x, \; t)}\right]}{n}$$

The age-specific growth rates can be inferred from the changes over time in numbers of deaths by age. Deaths in the prevailing life table (with radix one) are simply

$$d(a) = \frac{D(a)^{\int_o^a r(x)dx}}{\int_o^\infty D(a)e^{\int_o^a r(x)dx} da} \; .$$

Thus, from nothing more than two sets of age-specific numbers of deaths it is possible to construct a life table and to estimate birth rates (via equation 16). The required assumption is that mortality is constant during the interval of observation (and, of course, that the population is closed to or adjusted for migration). Since countries often collected and tabulated deaths by age before they conducted censuses, this procedure may find application in historical demographic research.

In this section and the succeeding one, it is assumed that the population is closed to migration, or, what is equivalent, that age-specific rates of net out-migration have been added to age-specific growth rates before the formulas are applied.

b) Birth rates and fertility

Estimating the birth rate from intercensal growth rates and a life table believed to prevail for the intercensal period can be done straightaway with equation (4). It is only necessary to substitute appropriate values into the equation. A particular advantage of this procedure is that it makes no use of the reported age distribution, which is often very seriously distorted at the young ages that are critical for many estimates of birth rates (e.g., through back-projection of age distributions). Instead, only age-specific growth rates are required, which would be unaffected by constant proportionate distortions at the first and second censuses. The age-specific growth rates could be combined with estimates of mortality made by Brass-type procedures based on reported numbers of children ever born and children surviving.

We have already shown how an estimate of the birth rate can be produced if the life table is unknown but is assumed to belong to a one-parameter set of model life tables, or if (not necessarily completely) registered deaths by age are available.

We also observed above that it is possible to estimate the net reproduction rate directly from the set of r(x)'s and the reported age distribution of mothers at childbirth. The proportion of births occurring to mothers aged a, v(a), at any time t is

$$v(a) = e^{-\int_o^a r(x)dx} \; p(a)m(a) \quad .$$

A survey question on births in the past year, or information facilitating the selection of a model fertility schedule, will provide an estimate of v(a). Then the net reproduction rate can be estimated by rearranging this expression and integrating.

$$NRR = \int_\alpha^\beta p(a)m(a)da = \int_\alpha^\beta v(a)e^{\int_o^a r(x)dx} \, da \; .$$

By its simplicity, what this expression (and certain earlier ones) seems to be telling us is that estimates of the net reproduction rate and the net maternity function are more readily and robustly inferred from age-specific growth rates than are either fertility or mortality conditions separately. This is analogous to relations among crude rates, since the crude rate of natural increase gives us directly the difference between crude birth and crude death rates but no separate information on either.

Armed with an estimate of the net reproduction rate, one can determine the approximate value of the gross reproduction rate (and the total fertility rate) by the use of two well-known approximations: NRR = GRR $p(\bar{m})$ (where $p(\bar{m})$ is the probability of surviving to the mean age of the net maternity function), and TFR = GRR (1+SRB), where SRB is the ratio of male to female births. The proportion surviving to $\bar{m}$ can be approximated from Brass-style estimates of $\ell(3)$ or $\ell(5)$ plus estimates of survival from childhood to $\bar{m}$ from some form of model life table, and 1+SRB can be taken as about 2.05. If the whole series of p(a) can be estimated, age-specific fertility rates can be estimated by $m(a) = v(a)\, e^{\int_0^a r(x)dx} / p(a)$.

Like other demographic series, age-specific growth rates are subject to error. When estimated from intercensal population change, they are subject to error from differences in coverage completeness between the censuses and from intercensal changes in the patterns of age misreporting. Age misreporting tends to have a large geo-culture component; patterns have apparently been very constant over a half century in India, for example (Zlotnik, 1979). Age tends to be quite well reported in countries of the Chinese-Japanese cultural sphere. There is usually little reason to expect that patterns of age misreporting will change radically from one census to the next, although the wording of age questions and instructions to enumerators can provoke such changes. If changes in the pattern of age misreporting involve only transfers between two adjacent age groups, the effect on the equations should not be large since they all involve the cumulative <u>sum</u> of growth rates up to a particular age.

Differences in census coverage completeness may be more problematic than the changes in age misreporting for most countries. A 2 percent improvement or deterioration in coverage between censuses separated by 10 years will evoke a change in all age-specific growth rates by .002. This is not a trivial magnitude in terms of its effect on the $\exp\{-\int_0^a r(x)dx\}$ function, which will change by the factor .951 by age 25. No single strategy can be enunciated for dealing with an erroneous series of growth rates. If all other demographic information is accurate, it is of course possible to estimate the error in the age-specific growth series directly by applying equation (5) to successive ages. This set of error estimates would then provide a direct way of correcting the second census to make it comparable in completeness and age misreporting to the first.

But it will be rare that other information can be assumed completely accurate. The general situation is one where nothing is known for certain. Here the new equations at least provide tests of consistency additional to those normally used. The most common consistency test compares estimated crude birth and death rates with recorded population growth from censuses. We can add to that test one in the form of equation (6) that displays a necessary relationship among age-specific growth rates and age-specific fertility and mortality rates prior to the end of childbearing. Because the Brass procedures for estimating age-specific mortality and fertility are widely used, opportunities for such an application are abundant. Equation

(5) is also a strong check of consistency among estimated birth rates, age-specific mortality, and age-specific growth.

It is also possible to estimate the degree of differential coverage in the two censuses, providing that one is willing to assume it to be invariant to age or to follow some other pre-specified functional form. If the second census is uniformly in error relative to the first by a ratio constant with age, then all computed age-specific growth rates will be in error by the same absolute amount γ. In the presence of such an error, all of the r(x)'s in formulas 4-6 must be replaced with $r_R(x) + \gamma$, where $r_R(x)$ is the observed (i.e., erroneous) growth rate at age $\underline{x}$. Equation (5) now becomes

$$c(a) = be^{-\int_0^a r_R(x)dx} e^{-\gamma a} p(a).$$

One may estimate γ by taking logs of both sides and rearranging:

$$\ell nc(a) - \ell np(a) + \int_0^a r_R(x)dx = \ell nb - \gamma a \ . \qquad (18)$$

The value of γ can now be estimated as the slope of a line. If registered deaths are available but the completeness of registration is an unknown, designated C as before, then

$$\ell nc(a) + \int_0^a r_R(x)dx = \ell nb - \gamma a - \frac{1}{c}\int_0^a \mu_R(x)dx \ . \qquad (19)$$

Equation (19) is now a linear equation with two independent variables that should not be highly colinear, so that identification of γ and C should be possible.

Still other procedures can be devised for use with model life table systems (e.g., Preston and Bennett, 1982). We cannot hope to be exhaustive here, and each of the procedures described needs much more careful attention to detail (e.g., treatment of open-ended age intervals) than we have provided. The new equations provide numerous fresh points of entry for demographic estimation, and we have only scratched the surface of possibilities as well as problems.

It should be noted that in virtually all of the measurement procedures described here, a corrected age distribution is an important by-product. The true age distribution of the population is itself an object of interest, and demographers can play a useful role in identifying it more accurately.

c) Migration

The conventional way to estimate net migration rates in the absence of a count of migrants is to forward project a population age distribution at time $\underline{t}$ by an "appropriate" life table and compare the projected population with that recorded at some time t+n (United Nations, 1970). Differences between actual and projected numbers of persons are ascribed to net surviving migrants. Back-projections of these survivors are then required in order to estimate the volume of net migration. The migration of persons who were below age n at time t+n requires special treatment. The procedure is awkward to implement unless censuses are separated by an integer multiple of five years because census age distributions are normally tabulated in five-year age categories.

A simple alternative is to use the equations for an open population. Since

$$N(a) = N(0)e^{-\int_0^a r(x)dx}\, e^{-\int_0^a e(x)dx}\, p(a),$$

$$\frac{N(a)e^{\int_0^a r(x)dx}}{N(0)p(a)} = e^{-\int_0^a e(x)dx}, \text{ and}$$

$$-\int_0^a e(x)dx = \ln \frac{N(a)}{N(0)p(a)} + \int_0^a r(x)dx. \qquad (20)$$

Implementing equation (20) again requires an "appropriate" life table to give p(a), plus census age distributions and age-specific growth rates. If implemented from age 0, it also requires intercensal births; if these cannot be estimated, the process could begin at age 5, with N(5) estimated by averaging numbers in the adjacent 5-year intervals. Applying equation (20) to successive ages gives the sum of age-specific net migration rates at different ages; age-specific net migration rates could then be estimated by subtraction. It is likely that imposing a "model" schedule of migration rates of the kind proposed by Rogers and Castro (1981) would improve estimates in developing countries. The procedure is clearly applicable to all forms of migration, whether internal (in which case the N(a)'s would pertain to a particular region of a country) or international. The advantage of using (19) relative to existing techniques is likely to be more of convenience than of methodological soundness. It does, however, provide an opportunity for improved estimates below age 10.

Estimates of Marital Survival

By analogy to previous results,

$$M(a) = M(0)e^{-\int_0^a r(x)dx}\, \pi(a), \text{ where} \qquad (21)$$

M(a) = number of marriages intact at duration a

r(x) = growth rate of number of married couples

π(a) = probability that a marriage will survive to duration a according to conditions of divorce and death of the period.

To estimate the life expectancy of a marriage from the time it was contracted according to period-specific conditions of dissolution, it is only necessary to rearrange this equation and integrate:

$$e_o(M) = \frac{\int_o^\infty M(a)e^{-\int_o^a r(x)dx}}{M(0)}$$

This provides a simple method of estimating the life expectancy of a marriage, which is otherwise so laborious a process that it is rarely undertaken. All that is required are two surveys giving the number of intact marriages by duration and an estimate of the number of intervening marriages that have occurred (M(0)). There are many other processes that could be similarly modelled: length of time spent in school, in prison, in parity two, in the divorced state, in the major leagues, in the priesthood, etc.

The above relationship does not indicate the likelihood of leaving the state of marriage from any of the multiple sources of exit. Now suppose that we have data on the number of divorces by duration of marriage, X(a). Multiplying both sides of (21) by $\mu^D(a)$, the force of decrement from divorce at duration <u>a</u>, we have

$$X(a) = M(a)\mu^D(a) = M(0)e^{-\int_o^a r(x)dx} \pi(a)\mu^D(a). \qquad (22)$$

The function, $\pi(a)\mu^D(a)$, integrated over all durations from 0 to ∞, is simply the probability that a marriage will end in divorce, p^D. Thus, rearranging (22) and integrating, we have

$$p^D = \frac{\int_o^\infty X(a)e^{\int_o^a r(x)dx} da}{M(0)} \qquad (23)$$

Equation (23) provides an extremely simple procedure for estimating the probability that a marriage will end in divorce. It generalizes one given in Preston (1975) that assumed stability. Again, it is widely applicable beyond the case of marriage and divorce. In the case of fertility, p^D is equivalent to a parity progression ratio, the probability of eventually leaving a particular parity by the route of having another child. With two surveys on the duration since achieving a particular parity (including zero) and the number of intervening births by order and duration since last birth, one can estimate all of the parity progression ratios and hence the total fertility rates without any reference to age. This generalizes some recent work of Griffith Feeney (1981).

The multiple decrement results pertain when duration in a state is the indexing variable. They are directly analogous to age relations in a population because one can only enter the duration hierarchy at zero, just as one enters the age hierarchy at birth. If one is interested in the expected years of life spent before the occurrence of some event, or the probability that some event will occur in the course of life, one would return to age as the indexing variable. Analogous versions of (23) exist, for example, to estimate the probability that an individual would marry, become a mother, enter the labor force, or move from place of birth. Only a slight modification is required to estimate the length of life before an individual enters one of these states.

Summary and Conclusion

Much of formal demography deals with functions that pertain to individuals passing through life, or, equivalently, to a stationary population in which the births of individuals are evenly distributed over time. These functions include life expectancy, probabilities of surviving between two ages, net and gross reproduction rates, expected years spent in various states, and the probability that particular events will occur in the course of life. The stable population model has proven very useful in part because it permits the translation of population structure or processes in a more general type of population — one with constant growth rates — back into equivalent functions for a stationary population. Here we have developed a method for translation that is more general still, since it applies to any population. The only ingredient required for the translation is a set of age-specific growth rates. These are also useful for performing the reverse translation, e.g., between a population's life table and its birth rate, or its age distribution.

Table 6 summarizes the basic relations among certain functions in a stationary population, a stable population, and any population. The $r(x)$ function used in the table is the age-specific growth rate plus the age-specific rate of net emigration. If the population is closed to migration, $r(x)$ is simply the age-specific growth rate. The meaning of the functions and variables has been previously defined.

Once the basic principle of this translation is recognized, its implementation becomes routine. We have described certain applications of the new equations, particularly to demographic estimation from incomplete data. The equations can be applied to many other issues: the two-sex problem, increment-decrement tables, convergence of a population to its stable form, cyclical changes in vital rates, and density dependence of population processes, to name a few. Stable population models will no doubt continue to occupy a central place in demonstrating the long-term implications of changes in mortality and fertility. However, in demographic estimation and measurement, it seems likely that the new procedures will supplant most of those based upon stable or quasi-stable assumptions. The existence of these procedures strongly underscores the value of repeated census operations for demographic measurement.

Table 6: Formulas for certain functions in stationary, stable, and any population.

Function	Notation	Formula for: Stationary Population	Formula for: Stable Population	Formula for: Any Population
Proportionate age distribution	$c(a)$	$bp(a)$	$be^{-ra}p(a)$	$be^{-\int_0^a r(x)dx}p(a)$
Ratio of population at two ages	$\frac{c(a+n)}{c(a)}$	nPa	$e^{-rn}nPa$	$e^{-\int_a^{a+n} r(x)dx}nPa$
Life expectancy at birth	$e_o^o = \int_0^\infty p(a)da$	$\frac{\int_0^\infty c(a)da}{b} = \frac{1}{b}$	$\frac{\int_0^\infty c(a)e^{ra}da}{b}$	$\frac{\int_0^\infty c(a)e^{\int_0^a r(x)dx}}{b}$
Birth rate	b	$\frac{1}{\int_0^\infty p(a)da}$	$\frac{1}{\int_0^\infty p(a)e^{-ra}da}$	$\frac{1}{\int_0^\infty p(a)e^{-\int_0^a r(x)dx}}$
Proportionate age distribution of mothers at childbirth	$v(a)$	$p(a)m(a)$	$p(a)m(a)e^{-ra}$	$p(a)m(a)e^{-\int_0^a r(x)dx}$
Net reproduction rate	$NRR = \int_\alpha^\beta p(a)m(a)da$	$\int_\alpha^\beta v(a)da = 1$	$\int_\alpha^\beta v(a)e^{ra}da$	$\int_\alpha^\beta v(a)e^{\int_0^a r(x)dx}da$
Expected years of life to be spent in state G with incidence at age <u>a</u> g(a)	$G'_L = \int_0^\infty g(a)p(a)da$	$\frac{\int_0^\infty g(a)c(a)da}{b}$	$\frac{\int_0^\infty g(a)c(a)e^{ra}da}{b}$	$\frac{\int_0^\infty g(a)c(a)e^{\int_0^a r(x)}da}{b}$
Number of persons at age a^* in terms of deaths above age a^*	$N(a^*)$	$\int_{a^*}^\infty D(a)da$	$\int_{a^*}^\infty D(a)e^{r(a-a^*)}da$	$\int_{a^*}^\infty D(a)e^{\int_{a^*}^a r(x)dx}da$
Number of persons at age a^* in terms of deaths below age a^*	$N(a^*)$	$N(0) - \int_0^{a^*} D(a)da$	$e^{ra^*}\left[N(0)-\int_0^{a^*}D(a)e^{ra}da\right]$	$e^{\int_0^{a^*} r(x)dx}\left[N(0)-\int_0^{a^*}D(a)e^{\int_0^a r(x)dx}da\right]$
Probability of survival from a^* to a^*+n in terms of deaths	nPa^*	$\frac{\int_{a^*+n}^\infty D(a)da}{\int_{a^*}^\infty D(a)da}$	$\frac{\int_{a^*+n}^\infty D(a)e^{r(a-a^*)}da}{\int_{a^*}^\infty D(a)e^{r(a-a^*)}da}$	$\frac{\int_{a^*+n}^\infty D(a)e^{\int_{a^*}^a r(x)dx}}{\int_{a^*}^\infty D(a)e^{\int_0^a r(x)dx}da}$

<u>Footnotes</u>

1. Calculated from Keyfitz and Flieger (1968, pp. 30-1, 230-2) and <u>Population Index</u>, April 1977, p. 374.

2. As Shiro Horiuchi has shown in correspondence, an expression for the age-specific growth rate itself, rather than its cumulation from age zero, can be derived by differentiating the second expression for N(a, t), giving

$$r(a,t) = r_B(t-a) - \int_o^a \frac{\partial\mu(x,y)}{\partial y}\,dx, \text{ where } y = t-a+x$$

$$\text{and } r_B(t) = d\ell n\, B(t)/dt.$$

3. More generally, if completeness varies with age, the left-hand side of (16) will equal a weighted mean value of age-specific completeness, where weights are supplied by the d(a) function, life table deaths at age *a*.

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Appendix

Derivation of the Basic Equation Linking Age Distributions to Period Mortality, Migration, and Growth Rates

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The proof of equation (3) is a straightforward application of multivariate calculus. What we present here is basically an expanded and elaborated version of an appendix in Bennett and Horiuchi (1981). Imagine a surface representing the number of persons alive by age and time period and define N(a, t) as the number of persons aged *a* at time *t*. The number of persons aged a + da at time t + dt is N(a + da, t + dt). For present purposes we will assume that da = dt, so that N(a, t) and N(a + da, t + dt) refer to persons belonging to the same cohort, i.e., those born at time (t - a). The change in the size of this cohort between time *t* and t + dt can be

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denoted as dN(a, t). Assuming existence and continuity of the partial derivatives, it can be shown that as da = dt approach zero,

$$dN(a, t) = \frac{\partial N(a, t)}{\partial t} dt + \frac{\partial N(a, t)}{\partial a} da \quad , \qquad (A.1)$$

where $\frac{\partial N(a, t)}{\partial t}$ is the partial derivative of N(a, t) with respect to t; and

$\frac{\partial N(a, t)}{\partial a}$ is the partial derivative of N(a, t) with respect to a.

Dividing both sides of A.1 by N(a, t), we have

$$\frac{dN(a, t)}{N(a, t)} = \frac{\frac{\partial N(a, t)}{\partial t} dt}{N(a, t)} + \frac{\frac{\partial N(a, t)}{\partial a} da}{N(a, t)} \qquad (A.2)$$

$$= r(a, t)dt + \frac{\frac{\partial N(a, t)}{\partial a}}{N(a, t)} da \ .$$

r(a,t) is the growth rate of the population aged a at time t, or the proportionate change in the number of persons aged a per unit of time. The left-hand side of A.2 is the proportionate change in the size of the cohort aged a at time t in the small interval of age a to a + da (or time t to t + dt). There are only two sources of change in a cohort's size, death and migration. Using ${}_{da}D(a)$ to denote deaths in the interval a to a + da to the cohort aged a at time t and ${}_{da}M(a)$ to denote net migrants (in-migrants minus out-migrants) during this same interval, we have

$$dN(a, t) = {}_{da}M(a) - {}_{da}D(a) \quad .$$

It is conventional to define the force of mortality function for a cohort at age a as (Keyfitz, 1968, p. 5)

$$\mu(a) = \lim_{da \to 0} \frac{{}_{da}D(a)}{N(a)da} \quad ,$$

where ${}_{da}D(a)$ is understood to pertain to the age interval a to a + da. We can analogously define the force of migration function as

$$\gamma(a) = \lim_{da \to 0} \frac{{}_{da}M(a)}{N(a)da} \quad .$$

Dividing both sides of A.2 by da = dt and substituting, we have,

$$\text{as } da = dt \to 0, \ -\mu(a, t) + \gamma(a, t) = r(a, t) + \frac{\frac{\partial N(a, t)}{\partial a}}{N(a, t)}. \qquad (A.3)$$

This is the equation linking ages, periods, and cohorts that is required in order to derive the remaining expressions.

A.3 can also be written as

$$\frac{\partial \ln N(a, t)}{\partial a} = \gamma(a, t) - \mu(a, t) - r(a, t) \quad .$$

Holding t constant and omitting it in the notation, we integrate both sides between specific ages 0 and x:

$$\int_0^x \frac{d\,\ell n\, N(a)}{da}\, da = \int_0^x \gamma(a)da - \int_0^x \mu(a)da - \int_0^x r(a)da, \text{ or}$$

$$\ell nN(x) - \ell nN(0) = \int_0^x \gamma(a)da - \int_0^x \mu(a)da - \int_0^x r(a)da \quad .$$

Taking exponentials and rearranging we have

$$N(x) = N(0)e^{\int_0^x \gamma(a)da} \; e^{-\int_0^x \mu(a)da} \; e^{-\int_0^x r(a)da} \quad .$$

This is the basic equation (7) in the text, with $\gamma(a)$ defined to equal $-e(a)$. In a closed population, of course, $\gamma(a) = 0$ at all a.

To develop the equivalent formulas for dealing with discrete time and age groups in a closed population, we return to equation A.3 and write it as

$$\mu(a, t) = -\frac{\frac{\partial N(a,\, t)}{\partial a}}{N(a,\, t)} - \frac{\frac{\partial N(a,\, t)}{\partial t}}{N(a,\, t)} \text{ , or}$$

$$D(a,\, t) = -\frac{\partial N(a,\, t)}{\partial a} - \frac{\partial N(a,\, t)}{\partial t} \quad .$$

We now integrate between specific ages x and x+n and periods t_1 to t_2:

$$\int_{t_1}^{t_2}\int_x^{x+n} D(a,\, t)dadt = -\int_{t_1}^{t_2}\int_x^{x+n} \frac{\partial N(a,\, t)}{\partial a}\, dadt - \int_x^{x+n}\int_{t_1}^{t_2} \frac{\partial N(a,\, t)}{\partial t}\, dtda$$

$$= -\int_{t_1}^{t_2}\{N(x+n,t)-N(x,t)\}dt - \int_x^{x+n}\{N(a,t_2)-N(a,t_1)\}da.$$

Now dividing both sides by the sum of person-years lived in the age and time interval, we have

$${}_nM_x = \frac{-d\ell n\left[\int_{t_1}^{t_2}\int_x^{x+n} N(a,\, t)dadt\right]}{dx} - {}_nr_x \text{, or}$$

$${}_nM_x = \frac{-d\ell n\, {}_nP_x}{dx} - {}_nr_x \quad .$$

The term inside the brackets, ${}_nP_x$, is the sum of person-years lived in the discrete time-age interval. ${}_nM_x$ is the death rate for that interval as conventionally defined: total deaths divided by total person-years lived. ${}_nr_x$ is the growth rate of the population in the interval as conventionally defined: the difference between the end period population aged x to x+n and the beginning period population in the age interval, divided by total person-years lived in the age interval during the period t_1 to t_2.

Now integrating this expression between specific ages 0 to K, we have

$$\int_0^K {}_nM_x dx = - \ln {}_nP_K + \ln {}_nP_0 - \int_0^K {}_nr_x dx \quad , \text{ or}$$

$$ {}_nP_K = {}_nP_0 \, e^{-\int_0^K {}_nM_x dx} \, e^{-\int_0^K {}_nr_x dx} \qquad (A.4)$$

This is the discrete analogue of equation (3) and the similarity is quite close. Person-years lived in discrete intervals of age and time have replaced N(a, t); mortality and growth defined on discrete age-time intervals have replaced $\mu(a, t)$ and $r(a, t)$. Note that the exp $\{-\int_0^K {}_nM_x dx\}$ term does not involve summing age-specific death rates in successive age intervals but rather requires summing death rates in n-year wide intervals continuously from starting ages 0 to K. This exponential term can be conveniently simplified by noting that

$$\frac{d}{dx} {}_nL_x = \ell_{x+n} - \ell_x = {}_nd_x .$$

Hence, $-{}_nm_x = \frac{(d \log_n L_x)}{dx}$; and on the assumption that

$$ {}_nM_x \doteq {}_nm_x , \; e^{-\int_0^K {}_nM_x dx} = {}_nL_K / {}_nL_o .$$

$$\text{Thus, } {}_nC_K = {}_nC_o \, e^{-\int_o^K {}_nr_x dx} \; {}_nL_K / {}_nL_o \quad . \qquad (A.5)$$

Note that A.5 shows the proportions at all age intervals (except the first) relative to ${}_nC_o$, which is in principle the first observation of grouped data. No precise relation to the number of births can be derived in this context. However, since

$$({}_nC_o + {}_nC_n + \ldots {}_nC_{\omega-n}) = 1.0,$$

it follows that

$$ {}_nC_o \left(1 + \frac{{}_nC_n}{{}_nC_o} + \ldots \frac{{}_nC_{\omega-n}}{{}_nC_o}\right) = 1.0 \quad . \qquad (A.6)$$

Since all of the terms but ${}_nC_o$ in A.6 can be calculated (when ${}_nL_x$ is known at n-year intervals, and when ${}_nr_x$ is known as a continuous variable), ${}_nC_o$ can be determined, and the other ${}_nC_K$'s as well. Note that (except for the generally accepted approximation that ${}_nm_x = {}_nM_x$) equation A.5 is exact. It is approximate only if ${}_nr_x$ itself is known at n-year intervals, rather than continuously. (See the calculations of the Swedish age distribution by five-year intervals as an illustration of this point.)

The relation between age structures of deaths and person-years in discrete time and age segments can be readily derived. Denoting ${}_nD_x$ as deaths in the age interval x to x + n during the time period t_1 to t_2, we have

$${}_nD_{x+y} = {}_nP_{x+y} \cdot {}_nM_{x+y} \ .$$

Substituting for ${}_nP_{x+y}$ from A.4,

$${}_nD_{x+y} = {}_nP_x \, e^{-\int_x^{x+y} {}_nM_a \, da - \int_x^{x+y} {}_nr_a \, da} \; {}_nM_{x+y}, \text{ or}$$

$${}_nD_{x+y} \, e^{\int_x^{x+y} {}_nr_a \, da} = {}_nP_x \, e^{-\int_x^{x+y} {}_nM_a \, da} \; {}_nM_{x+y} \ .$$

Integrating both sides of this expression from y=0 to y = ∞, we have

$$\int_0^\infty {}_nD_{x+y} \, e^{\int_x^{x+y} {}_nr_a \, da} \, dy = {}_nP_x \int_0^\infty e^{-\int_x^{x+y} {}_nM_a \, da} \; {}_nM_{x+y} \, dy \ .$$

But the value of the integral on the right-hand side is unity, as can be shown by integrating by parts. Therefore,

$${}_nP_x = \int_0^\infty {}_nD_{x+y} \, e^{\int_x^{x+y} {}_nr_a \, da} \, dy \ .$$

This equation is exactly analogous to one in the text except that ${}_nP_x$ has replaced N(a), ${}_nD_x$ has replaced D(a), and ${}_nr_x$ has replaced r(a).