Demography

Measuring and Modeling Population Processes

Samuel H. Preston
Patrick Heuveline
Michel Guillot

BLACKWELL Publishers

First published 2001

2 4 6 8 10 9 7 5 3 1

Blackwell Publishers Ltd
108 Cowley Road
Oxford OX4 1JF
UK

Blackwell Publishers Inc.
350 Main Street
Malden, Massachusetts 02148
USA

British Library Cataloguing in Publication Data
A CIP catalogue record for this book is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Preston, Samuel H.
Demography: measuring and modeling population processes / Samuel H. Preston, Patrick Heuveline, Michel Guillot.
p. cm.
Includes bibliographical references and index.
ISBN 1-55786-214-1 (alk. paper)—ISBN 1-55786-451-9 (pb: alk. paper)
1. Demography. 2. Population research. I. Heuveline, Patrick. II. Guillot, Michel. III. Title.

HB849.4 .P73 2000
304.6′07′2—dc21 00-033721

Typeset in 10 on 12 pt Times Roman
by Newgen Imaging Systems (P) Ltd, Chennai, India
Printed in Great Britain by TJ International Ltd, Padstow, Cornwall

This book is printed on acid-free paper.

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Alberto Palloni, University of Wisconsin

12.1 Introduction

In chapter 3 we studied the life table as a tool to describe mortality. A life table can be used to describe *any* event whereby individuals under observation transit from one state to another. In the case of mortality the event is death and the two states under consideration are "alive" (state 1) and "dead" (state 2).

The mortality process studied in chapter 3 can be thought as representing a model resting on the following assumptions:

a. *Simple state space*: There are only two possible states that individuals can occupy;
b. *Event is proper*: All individuals eventually transit from state 1 to state 2;
c. *Destination state is "absorbing"*: Nobody who moves from state 1 to state 2 ever goes back to state 1.

Most demographic phenomena consist of events that cannot be represented and comprehended by such a basic model. To represent the marriage process, for example, we need to modify assumption (a) by increasing the number of possible states to include single, married, widowed, and divorced. These states are clearly not absorbing because people who enter them may subsequently leave them.

Chapter 4 discussed modifications to the simple life table procedure to handle one important generalization, namely, the introduction of multiple and competing destination states. This is an extension that removes assumption (a). However, the multiple decrement model continues to be restrictive since it relies on the other two assumptions, namely, that all destination states are absorbing (no reverse flows are possible), and that the events are proper, that is, everyone will experience the event under study or, equivalently, everybody will exit from state 1. Of these assumptions, that preventing reverse flows is most problematic for demographic computation. Section 12.2 describes increment–decrement life table models that enable us to understand events with non-absorbing states and reverse flows. Section 12.3 introduces an example,

discusses computational choices for the calculations to estimate an increment–decrement table, and suggests interpretations for the results. In the remaining sections of the chapter we formalize and generalize relations between quantities in any increment–decrement table.

12.2 Increment–Decrement Life Tables

In this section we review examples of three phenomena that can be fruitfully analyzed with generalized increment–decrement tables.

12.2.1 Marriage and divorce

The process of union formation and dissolution is the prime example of interrelated events that can be understood with a simple increment–decrement model. To keep the illustration simple we will neglect the existence of consensual unions and assume that all unions are formally sanctioned. We will also overlook equally important complexities raised by the fact that union formation and dissolution involve not just one but two individuals. In what follows we focus on women exposed to marriage.

In most populations a majority of, but not necessarily all, women will eventually marry. Some but not all among those who marry will experience a divorce (permanent separation) or widowhood due to the death of their partner. Finally, some marriages will be terminated as a consequence of the death of the woman herself. The multistate representation of these events is graphically displayed in figure 12.1 (Schoen, 1988). Women who marry for the first time cannot return to the single state, and thus there is only a one directional arrow linking the state "single" with the state "married". By contrast, those who divorce or separate and those who experience widowhood may remarry, and this possibility is reflected by two-directional arrows. As always, death is an absorbing state and there are no reverse flows from the state, "dead."

In this representation, the passage of time is measured by the age of the woman and one does not necessarily need to account for the time spent in each state as an important dimension of the problem. That is, the model assumes that remarriage and divorce depend on time only through the age of the woman and not through the duration that they have spent in any state. If this assumption is violated, special procedures to handle both age and duration dependence are needed.

Of interest to those studying marriage change and family formation are questions such as the following: what is the expected time before the first marriage? What is the probability

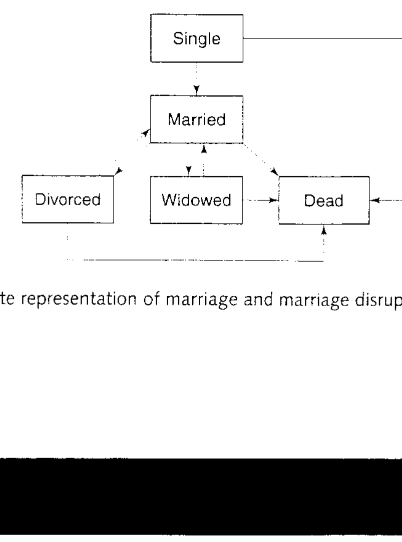

Figure 12.1 Multistate representation of marriage and marriage disruption

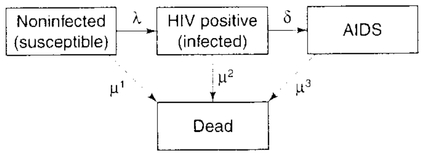

Figure 12.2 Multistate representation of HIV/AIDS

that a first marriage will eventually end up in a divorce? What is the average number of marriages that a woman will experience during her lifetime? What is the expected number of years before a first marriage breaks up by divorce? Answers to these questions may enable investigators to understand better the effects of social and economic forces on family formation and organization.

12.2.2 HIV/AIDS

Because of its very nature, the HIV epidemic admits a simple multistate representation (Palloni, 1996). Individuals in a population exposed to HIV can occupy one of three states: susceptible (noninfected), HIV-positive and asymptomatic (contracted virus but with no symptoms of AIDS), and AIDS (full-blown symptoms of AIDS). As always, death is an absorbing state. A graphic representation appears in figure 12.2. The force of infection, λ, is the instantaneous rate of infection or HIV incidence; the force of incubation, δ, is the instantaneous rate of incubation or AIDS incidence; and the quantities μ^i ($i = 1, 2, 3$) are, respectively, the forces of mortality for individuals who are susceptible, infected, and with AIDS. As in any application of life table procedures, our interest is to use observed events, namely, becoming infected, developing AIDS, and dying, to estimate the underlying rates, λ, δ, and μ^i.

This example shares an important feature with simple life table representations: there are no reverse flows, as individuals who become infected will remain infected for life. As before, death is an absorbing state. However, not everybody in the population is likely to become infected. Indeed, an important quantity to be estimated is the ultimate proportion of individuals who are likely to become infected.

In the case of HIV/AIDS the issue of time dependence is more complicated than in the case of marriage. Indeed, while the force of infection is mainly dependent on the age of individuals, the force of incubation is driven by the duration in the state (duration of infection) as much as it is by age. By the same token, the risk of mortality once AIDS is contracted, μ^3, is almost entirely associated with duration of infection and only weakly dependent on age.

As in the case of marriage, an understanding of the HIV/AIDS epidemic requires us to answer questions that increment–decrement tables can address very precisely: what proportion of a cohort will eventually contract the virus? What is the average age at which individuals in a cohort will contract the virus? What fraction of the cohort will contract the virus before reaching age x? What fraction of the cohort will develop AIDS before age y?

12.2.3 Health, chronic illnesses, and disability

An important and lively current debate in the study of health and mortality revolves around the idea that as improvements in survivorship and life expectancy continue, the health of those

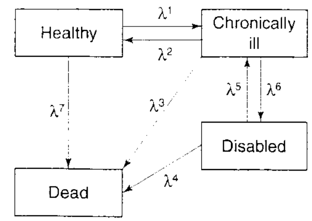

Figure 12.3 Multistate representation of chronic illness and disability

individuals benefited by these improvements may deteriorate (Fries, 1980; Singer and Manton, 1994). Is it generally true, for example, that the time spent ill or disabled is longer now than it was when life expectancy was around 60 years? Could it be that when they reach their retirement age baby boomers will experience higher life expectancy but also higher prevalence of ill-health and disability than their parents did at similar ages? If this is so, what kind of pressure will there be on resources to keep a minimum standard of well-being?

A simple way to understand the events and relations involved and indeed to begin to answer these questions is, once again, through an increment–decrement representation (Rogers et al., 1990). This is shown in figure 12.3. We assume that all individuals start out in the state "healthy" and that they can transit first to the state "chronically ill" and from there to the state "disabled." As most disability is caused by chronic illnesses, we will neglect the possibility of a flow from "healthy" to "disabled." As in the case of marital status, reverse flows are possible as individuals can recover from either disability or chronic illnesses.

Strategic factors that will shape the answers to the questions formulated before are the set of transition rates, λ^1 and λ^2. These are, respectively, the rate of incidence of chronic illness and the rate of recovery. To the extent that λ^1 remains invariant over time but λ^2 decreases, we should expect that a growing fraction of the population will be occupying the state "chronically ill" or "disabled." Note also that if λ^3 or λ^4 are reduced (death rates fall among the ill and disabled) and all the other rates remain unchanged, we should expect a similar result, namely, an increase in the prevalence of chronic illness and disability. Understanding the factors that determine these rates is then a key to providing evidence for or against the idea that morbidity is increasing or expanding.

There are a number of other examples and illustrations that we could have used. Paramount among these are applications to multiregional life tables, where the analyst is interested in modeling migration flows between and within geographic regions as well as mortality (Rogers, 1995b).

12.3 Estimation of Increment–Decrement Life Tables

12.3.1 Children's experiences of marriage, consensual unions, and disruptions

A controversial topic in the current demographic and sociological literature is related to the changing dynamic of union formation and dissolution. Over the past twenty years or so the rate of entrance into consensual unions has increased sharply. This is thought by some to be responsible for increasing rates of childbearing out of wedlock. In addition, some researchers believe that consensual unions are more likely to end up in eventual separation and that, even if they lead to a marriage, the latter is subject to an increased chance of divorce. These transformations certainly influence the family structure and material well-being of couples, but are thought to produce particularly salient consequences for the early experiences of children. Because children's early life experiences have potentially large effects on their later life behaviors and activities, it becomes important to describe children's patterns of exposure to different types of family contexts dictated by their parents' union history.

We can shed some light on this issue by summarizing the experiences of children at various ages as a function of one (or both) of their parents' union status (see Bumpass and Lu, 2000). For simplicity, we choose to work with their mother's union status. Thus, the study population consists of children whose mother's union history will determine the family context which children encounter at a given point in their life. Since the most strategic issues associated with a child's living arrangements have to do with early life impact, we are justified to focus our attention on children's experiences between exact ages zero and 15. Similarly, since the main hypotheses suggest that the most important contrasts are associated with children who experience life with a single mother, with a mother in a consensual union, or with a mother in a marriage, we will neglect altogether all states created by mortality. As a consequence we start with a simplified representation of the marriage process displayed in figure 12.4. In this figure the states are numbered sequentially and the transition rates to and from any of them are indexed so that the first superscript always corresponds to the state of origin and the second to the state of destination. The corresponding rates for these transitions, $\lambda^{ij}(x)$, are associated with children, not with adult women and/or men, and refer to the rate at which children whose mothers are in state i at age x move to state j in the small age interval $(x, x+\delta x)$. Thus, $\lambda^{12}(x)$ is the rate at which children who live with mothers who are not in a union experience a change between ages x and $x+\delta x$ and begin to live with mothers who are cohabiting but not married. Similarly, $\lambda^{23}(x)$ is the rate at which children who live with a mother in a consensual union move between ages x and $x+\delta x$ to a family context characterized by a mother who is married.

Relative to the marital status example given before, the state-space representation in figure 12.4 is both simpler and more complicated. The illustration is more complicated because we now explicitly consider the existence of consensual unions as different from marriages. This complication is justified by the increasing importance of consensual unions and the increased prevalence of children who live with cohabiting parents who are not legally married.

The state-space representation, however, is also simpler since we neglect altogether the effects of mortality. In fact, not only have we omitted an absorbing state for death (of the child), but we also overlook the distinction, for example, between a child whose mother is not in a union due to death of a spouse or partner (widowhood) and a child whose mother is not

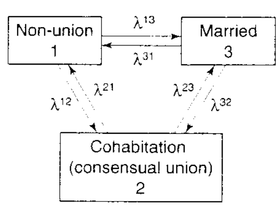

Figure 12.4 Multistate representation of children's experience with family contexts

in a union due to a divorce or separation. This decision is probably inconsequential since we are only interested in the evolution of the phenomena in a range of maternal ages (15 to 55) where adult mortality is very low. Thus, the rates λ^{21} and λ^{31} will reflect primarily the risk of separation (from consensual unions) and divorce (from marriage). By the same token, we do not distinguish among women in the non-union state according to the nature of their previous union. Instead, we lump together divorced mothers with those who were in a cohabiting union. This is tantamount to assuming that any heterogeneity in the transition rates out of this state (non-union) can be legitimately ignored or, equivalently, that the processes of union or marriage formation for those whose previous union was a cohabitation is essentially the same as it is for those whose previous union was a marriage. If this were not a realistic assumption – and, in all likelihood, it is not so – we should distinguish the existence of two disruption states.

12.3.2 Estimation of rates

The National Survey of Family Growth (NSFG) is a nationally representative survey of households in the United States implemented periodically by the National Center for Health Statistics. The goal of the survey is to retrieve information on fertility and related health issues. The NSFG-5 fielded in 1995 includes 10,847 female respondents who are 15–44 years of age in 1995 (Potter et al., 1997). Since the NSFG-5 elicits union and fertility histories for the women in the sample, we are able to reconstruct their children's experience of cohabitation, marriage, or union disruption (Bumpass and Lu, 2000). With the retrospective information on events that occurred during the period 1990–4, we calculate observed single-year age-specific rates for every relevant age and flow displayed in figure 12.4. These rates, which we will denote ${}_1M_x^{ij}$, correspond to the ratios ${}_1D_x^{ij}/{}_1N_x^i$ of observed transitions from i to j in the age interval x to $x+1$ (${}_1D_x^{ij}$) to the estimated midperiod population in state i in the age group (${}_1N_x^i$). These rates are displayed in table 12.1.[1] Just as the mortality rates ${}_1M_x$'s defined in chapter 3 were the basis for the life tables in the two-state case, so the ${}_1M_x^{ij}$'s rates will be the basis for life tables associated with each state in figure 12.4. Thus, in this application we will have one life table for children whose mothers are not in a union (state 1 or "non-union"), one for children whose mothers are in a consensual or cohabiting union (state 2, "cohabitation"), and one for children whose mothers are married (state 3, "married"). As in chapter 3, the central quantities in these tables are the corresponding probabilities of experiencing the events.

Recall now the procedure used when there are only two states to consider, origin and destination, and only one flow from origin to destination. In this case we only focused on quantities describing the exits or the flow away from the origin state. To do so we calculated values of ${}_1M_x$'s and added an assumption about the behavior of the underlying risk to estimate the values of ${}_1q_x$, ${}_1d_x$, and ${}_1L_x$. For example, with the assumption that $l(x)$ is linear over one-year intervals, we could uniquely estimate ${}_1q_x$ from ${}_1M_x$ for every age group, and then derive the quantities ${}_1d_x$ and ${}_1L_x$.[2] Although we did not describe the procedure quite in these terms, we could say that for each age (other than age 0) in the life table we have three equations in three unknowns. The three equations in the linear case are:

$$
\begin{aligned}
l(x+1) &= l(x) - {}_1d_x \\
{}_1d_x &= {}_1M_x \cdot {}_1L_x \\
{}_1L_x &= .5 \cdot [l(x) + l(x+1)]
\end{aligned}
\qquad \textbf{(12.1)}
$$

Table 12.1: *Observed values of* ${}_1M_x^{ij}$ *for the states and flows represented in figure* 12.4

Age	*State at beginning of age interval*					
	1 *(non-union) Destination*		2 *(cohabitation) Destination*		3 *(marriage) Destination*	
	2	3	1	3	1	2
0	.0777	.0421	.0968	.1460	.0121	.0086
1	.0858	.0405	.0984	.1411	.0211	.0082
2	.1068	.0350	.0759	.1468	.0196	.0069
3	.1054	.0354	.0829	.1639	.0210	.0045
4	.0832	.0475	.0656	.1282	.0216	.0084
5	.0939	.0497	.0555	.1433	.0214	.0076
6	.0617	.0469	.0506	.1229	.0251	.0022
7	.0808	.0580	.0471	.1326	.0201	.0078
8	.0507	.0305	.0655	.1387	.0196	.0027
9	.0621	.0375	.0815	.1430	.0215	.0031
10	.0854	.0411	.0508	.1370	.0201	.0049
11	.0435	.0343	.0855	.1149	.0186	.0032
12	.0656	.0521	.0880	.0896	.0260	.0043
13	.0427	.0313	.0812	.1307	.0204	.0071
14	.0837	.0314	.0851	.0712	.0260	.0066

Source: NSFG-5. See also Bumpass and Lu, 2000.

Since for each age x, $l(x)$ is known – a result of the recursion that starts from an arbitrary radix or value for $l(0)$ – the three unknowns are $l(x+1)$, ${}_1d_x$ and ${}_1L_x$. One can easily verify that the solution for ${}_1d_x$ is

$$ {}_1d_x = l(x) \cdot \frac{{}_1M_x}{1 + .5 \cdot {}_1M_x} $$

or, equivalently,

$$ l(x+1) = l(x) \cdot \frac{1 - .5 \cdot {}_1M_x}{1 + .5 \cdot {}_1M_x} \tag{12.2} $$

This implies that

$$ {}_1q_x = \frac{{}_1M_x}{1 + .5 \cdot {}_1M_x}, $$

the kernel of the empirical solution to the two-state life table in chapter 3.

We can proceed in an analogous way in the multistate case provided that we account for the fact that at each age there could be more than one flow. To do this most efficiently, it is convenient to introduce more notation. We define the following quantities:

$l^i(x)$ is the number of individuals in state i at exact age x;

${}_1d_x^{ij}$ is the number of individuals moving from state i to state j between ages x and $x+1$;

${}_1L_x^i$ is the number of person-years lived in state i between ages x and $x+1$.

The reader should verify that the following equalities hold for all ages:

$$ l^i(x+1) = l_i(x) + \sum_j {}_1d_x^{ji} - \sum_j {}_1d_x^{ij} \qquad \textit{Equation of type I} $$
$$ {}_1d_x^{ij} = {}_1M_x^{ij} \cdot {}_1L_x^i \qquad \textit{Equation of type II} \tag{12.3} $$
$$ {}_1L_x^i = .5 \cdot [l^i(x) + l^i(x+1)] \qquad \textit{Equation of type III} $$

Since in the representation of figure 12.4 we have three different states, for each age group we will need three equations of type I, and also three equations of type III. Similarly, since there is a total of six flows or transitions, we will need six equations of type II. This amounts to a total of twelve equations. The unknowns will be three values of $l^i(x+1)$, three values of ${}_1L_x^i$ and six values of ${}_1d_x^{ij}$ or, equivalently, six values of the conditional probabilities ${}_1q_x^{ij}$.[3]

To generate estimates of the quantities we seek, ${}_1q_x^{ij}$, we need to solve a system of twelve equations in twelve unknowns for each age. This is certainly not a trivial task but it is not intrinsically difficult. Indeed, and as we show below, calculation of the solution involves consecutive operations of inversion of a matrix, one for each age group (except for age zero). Although the matrix inversion operation is not always smooth, there are a number of software packages that can handle the assignment very efficiently (see section 12.7).

Using the rates displayed in table 12.1 we solve the system of equations, one for each age group, and then proceed to calculate the quantities $l^i(x)$, ${}_1d_x^{ij}$, and ${}_1L_x^i$ for all three relevant states. All calculations are based on a radix of $l^i(0) = 1,000$ for all i, that is, we arbitrarily assume that we start with a synthetic cohort of 3,000 children aged 0, one thousand in each state. For accurate estimates, of course, it would be necessary to know the actual distribution of children at birth among the states.

Table 12.2 displays the values of ${}_1q_x^{ij}$, and table 12.3 displays the values of $l^i(x)$ and ${}_1d_x^{ij}$. In table 12.2 the values in the first of the three columns associated with each state correspond to the conditional probabilities of remaining in that state at the end of each one-year interval. Thus, in the first age group and for state 1, the value in the first column is .8902 (= 1 − .0657 − .0441). Although the values of ${}_1L_x^i$ are implicit in table 12.3, we omit them to avoid excessive cluttering. Finally, the estimated expected durations at age zero spent in each of the three states are displayed in table 12.4.[4]

12.3.3 Interpretation of estimates

For each origin state i and for each age x, the conditional probabilities of moving from state to state, ${}_1q_x^{ij}$, displayed in table 12.2 add up to 1.0, as they should. Thus, for example, the three possible transitions for a child aged zero who lived with a mother not in a union are: (a) to continue to experience the same living arrangement with probability .8902, or (b) to live with a cohabiting mother with probability .0657 or, finally, (c) to live with a married mother with probability .0441. Combining the conditional probabilities in table 12.2 with radices $l^i(0) = 1,000$ for each i leads to the figures displayed in table 12.3. According to this table, at exactly age 5 there are 837 children in state 1, 604 in state 2, and 1,559 in state 3. Note that these numbers add up to 3,000 since there is no child mortality in our representation, and all 3,000 children in the original cohort must be in one of the three states at every age.

The columns ${}_1d_x^{ij}$ also have straightforward interpretations. Thus, reading down the column ${}_1d_x^{12}$ we find that among the 837 children who lived with a mother not in a union at exact

Table 12.2: *Estimated values of* ${}_1q_x^{ij}$ *for the states and flows represented in figure* 12.4

Age	*State at beginning of age interval*								
	1 (*non-union*) *Destination*			2 (*cohabitation*) *Destination*			3 (*married*) *Destination*		
	1	2	3	1	2	3	1	2	3
0	.8902	.0657	.0441	.0823	.7868	.1308	.0117	.0080	.9802
1	.8850	.0724	.0427	.0841	.7898	.1261	.0199	.0080	.9721
2	.8711	.0900	.0381	.0651	.8032	.1317	.0183	.0070	.9747
3	.8723	.0879	.0398	.0705	.7839	.1456	.0196	.0049	.9756
4	.8800	.0715	.0485	.0574	.8259	.1167	.0202	.0083	.9714
5	.8686	.0800	.0514	.0485	.8217	.1298	.0199	.0077	.9724
6	.8988	.0540	.0472	.0455	.8418	.1127	.0236	.0027	.9738
7	.8723	.0696	.0581	.0416	.8371	.1213	.0187	.0078	.9735
8	.9237	.0443	.0320	.0583	.8162	.1254	.0187	.0029	.9784
9	.9076	.0533	.0391	.0712	.8003	.1285	.0204	.0033	.9763
10	.8832	.0736	.0432	.0449	.8304	.1247	.0188	.0052	.9760
11	.9270	.0381	.0349	.0758	.8195	.1047	.0178	.0032	.9789
12	.8918	.0571	.0511	.0775	.8394	.0832	.0244	.0046	.9710
13	.9304	.0374	.0322	.0720	.8102	.1178	.0197	.0067	.9736
14	.8945	.0736	.0318	.0756	.8582	.0663	.0245	.0070	.9685

Source: NSFG-5. See also Bumpass and Lu, 2000.

Table 12.3: *Estimated values of* $l^i(x)$ *and* ${}_1d_x^{ij}$ *for the states and flows represented in figure* 12.4

Age	*State i*								
	$i = 1$ (*non-union*)			$i = 2$ (*cohabitation*)			$i = 3$ (*married*)		
	$l^1(x)$	${}_1d_x^{12}$	${}_1d_x^{13}$	$l^2(x)$	${}_1d_x^{21}$	${}_1d_x^{23}$	$l^3(x)$	d_x^{31}	d_x^{32}
0	1,000	66	44	1,000	82	131	1,000	12	8
1	984	71	42	861	72	109	1,155	23	9
2	966	87	37	760	49	100	1,274	23	9
3	915	80	36	706	50	103	1,378	27	7
4	875	63	42	641	37	75	1,484	30	12
5	837	67	43	604	29	78	1,559	31	12
6	787	43	37	576	26	65	1,637	39	4
7	772	54	45	531	22	64	1,696	32	13
8	728	32	23	512	30	64	1,761	33	5
9	736	39	29	454	32	58	1,810	37	6
10	737	54	32	409	18	51	1,854	35	10
11	704	27	25	403	31	42	1,893	34	6
12	717	41	37	363	28	30	1,920	47	9
13	714	27	23	355	26	42	1,931	38	13
14	728	54	23	327	25	22	1,945	48	14
15	724	—	—	348	—	—	1,928	—	—

Source: NSFG-5. See also Bumpass and Lu, 2000. Calculated from table 12.2. Note that the sum of $l^i(x)$ for each row should be equal to 3,000. Discrepancies are due to rounding.

Table 12.4: *Expected duration (or waiting time) by state and state of origin*

State of origin (at age 0)	*Expected number of years to be lived in state j* =		
	1 (*non-union*)	2 (*cohabitation*)	3 (*married*)
All	4.0	2.7	8.3
1	8.0	2.5	4.6
2	2.8	5.3	6.9
3	1.3	.6	13.1

Source: First row calculated from table 12.3. Row entries do not always add to 15.0 due to rounding.

age five, 67 experience a change and begin living with a mother who is in a cohabiting union. Similarly, 43 of the original 837 children who were in state 1 at exact age 5 begin living with a married mother between their fifth and sixth birthdays. A similar interpretation applies to the other columns.

The functions $l^i(x)$ are not always monotonically decreasing, reflecting the fact that at every age and for each state there are both decrements and increments. Thus, it should be clear that it is not possible to use $l^i(x)/l^i(0)$ as a measure of the probability, for a newborn in state i, of remaining in state i at age x. Similarly, the ratios $l^i(x+k)/l^i(x)$ no longer measure the conditional probabilities of remaining in state i. One can, however, use the ratios

$$l^i(x) \Big/ \sum_i l^i(0)$$

to represent the probability that a newborn will *be* in state i at age x. For example, the probability that a newborn will live with a married mother at age 10 is .618 (=1854/3000).

There are two types of life expectancy, or expected waiting times, that can be derived from increment – decrement life tables. The first is an *unconditional* expected duration representing the average duration of time lived in a particular state, regardless of origin. According to table 12.4, the expected number of years spent in states 1, 2, and 3 by the members of our fictitious cohort are respectively 4.0 years, 2.7 years, and 8.3 years. This means that a member of the original cohort (regardless of his/her starting state) is expected to live 4.0 years of his life between ages zero and 15 (exactly) with a mother who is not in a union, 2.7 years with a mother in a cohabiting union, and the remaining years with a mother in a marriage. These figures add up to 15.0; we have accounted fully for all of the first 15 years of life.

The second kind of waiting time or duration in a state is called "conditional," and it is important to understand the difference between unconditional and conditional waiting times or duration (Schoen, 1988). The unconditional duration or life expectancy in state j at age x, $e^j(x)$ – the number of years of life to be lived in state j after age x – can be directly calculated from the values of ${}_1L_y^j\ (y \geq x)$ implicit in table 12.3. By construction these values are additive. In particular,

$$\sum_j e^j(0) = e(0)$$

In our case this is fifteen years since there is no child mortality.

By contrast, the conditional expectations or conditional duration, $\Psi^{ij}(x)$ – the expected number of years to be lived in state j by those who are in state i at exact age x – must be estimated using the trajectory followed by members of the cohort who occupy state i at exact age x and then calculating the time spent in state j. These calculations are tedious but not intrinsically difficult.

Suppose that there are $l^i(x)$ children who are in state i at exact age x. We then estimate a new set of life tables for a cohort with $l^i(x)$ members who start in state i and for whom the starting age is not zero but age x. From these newly estimated life tables we will obtain unconditional expectations or values $e^j(y)$ for all j and for $y \geq x$. These values are associated with the $l_i(x)$ children in the new initial cohort, not with the original cohort of children. To avoid confusion we will label these unconditional expectations $\epsilon^j(y)$ for $y \geq x$. It follows that the quantities we seek, $\Psi^{ij}(x)$, are indeed the values $\epsilon^j(x)$. It also follows that the total number of years lived after exact age x by those who are in state i at age x must equal

$$\sum_j \psi^{ij}(x)$$

The values displayed in the first row of table 12.4 correspond to $e^j(0)$ whereas those in the remaining rows are the quantities $\Psi^{ij}(0)$.[5] An interesting feature revealed by these quantities is that children born to a married mother will spend most of their first fifteen years of life in such a state, whereas those born to mothers not in a union or in a cohabiting union will spend much of their first fifteen years of life in those same states.

12.4 Formalization and Generalization of Relations

We now examine more closely the nature of the functions $l^j(x+1)$ and ${}_1d_x^{ij}$ and explore key interrelations between them.

12.4.1 The nature of ${}_1d_x^{ij}$

What is ${}_1d_x^{23}$ in our illustration? It is the number of children who reached exact age x with a mother in a cohabiting union and whose mother was married when they (the children) reached age $x+1$. This quantity is a result of a multiplicity of flows, some involving only one transition, others involving more than one transition, some involving moves away from state 3 and others moves into state 3. For example, it includes children whose mothers were in a union at age x and then married at age $x+\delta$ ($0< \delta <1$) and stayed married until age $x+1$. But it also includes children whose mothers experience a potentially more complicated sequence of moves such as: cohabiting when the child is age x, marrying when the child is $x+\delta$ ($0< \delta <1$), divorcing when the child is $x+\delta'$ ($\delta < \delta'$), entering another cohabiting union at age $x+\delta''$ ($\delta < \delta''$) and, finally, marrying again and staying married until the child reaches age $x+1$. Thus, ${}_1d_x^{23}$ is affected not just by flows *into* state 3 but also by those *out* of it. The quantity, therefore, *excludes* individuals who start in state 2, move to 3, and then exit this state without reentering it before attaining age $x+1$. It also excludes individuals with multiple transitions, such as the one described before but with an extra transition out of state 3 without reentry before age $x+1$. It is clear then that if rates of exits *out* of state 3 were lower, the quantity ${}_1d_x^{23}$ could be higher.

In most applications either the time intervals are very short or the rates are so low that the likelihood that an individual will experience multiple events in a single time interval is remote. But even in the most conservative case, that is, when only one move per individual per time interval is permitted, ${}_1d_x^{ij}$ is no longer to be considered a measure of pure decrements, except when j is an absorbing state.

12.4.2 The nature of $l^i(x+1)$

We already noted that the function $l^i(x+1)$ in table 12.3 is not strictly decreasing with age. For example, the value attained by $l^3(x)$ at age 7 (=1696) represents an increase relative to the value of $l^3(6)$ (=1637). This occurs because $l^3(x)$ reflects the ebb and flow of marriage as well as of the other phenomena. In the case of state 3, between ages six and seven there is a decrement of about 39 caused by divorces, and another decrement of 4 caused by transitions toward cohabitation. But there are also increments of 37 and 65 accounted for by transitions from state 1 and state 2 respectively. Clearly, the function $l^i(x+1)$ is influenced by the magnitude of the transition rates (into and out of the state) and also by the magnitude attained by the functions $l^j(x)$ for all j different from i.

12.4.3 General linkages

These linkages between the various states are rendered more fully if we construct a matrix containing the sources of *increments* and *decrements* for each state i. This matrix, which we call $\boldsymbol{l}(x+1)$, contains as elements the values ${}^il^j(x+1)$, the number of individuals who were in state i at exact age x and end up in state j by age $x+1$. For $i \neq j$ the function ${}^il^j(x+1)$ is equivalent to ${}_1d_x^{ij}$; indeed, these values represent the number of individuals who move from state i to state j in the age interval. By contrast, the values of the function ${}^il^i(x+1)$ represent the survivors of the original "cohort" of individuals who started out in state i at age x. Therefore it must be a strictly decreasing function of age that depends on the initial value ${}^il^i(x)$ on the one hand, and on the decrements consisting of all those individuals who moved out of state i, namely

$$\sum_{j \neq i} {}_1d_x^{ij},$$

on the other.

We can now establish the link between the elements of $\boldsymbol{l}(x+1)$ and those of $\boldsymbol{l}(x)$ via the quantities ${}_1d_x^{ij}$. In a very general situation when we have non-absorbing states $i = 1, \ldots, k$:

$$\boldsymbol{l}(x+1) = \boldsymbol{l}(x) - \boldsymbol{D}(x) \qquad \textbf{(12.4)}$$

or, in longhand:

$$\begin{pmatrix} {}^1l^1(x+1) & {}^1l^2(x+1) & \cdots & {}^1l^k(x+1) \\ {}^2l^1(x+1) & {}^2l^2(x+1) & \cdots & {}^2l^k(x+1) \\ \cdots & \cdots & \cdots & \cdots \\ {}^kl^1(x+1) & {}^kl^2(x+1) & \cdots & {}^kl^k(x+1) \end{pmatrix} = \begin{pmatrix} {}^1l^1(x) & 0 & \cdots & 0 \\ 0 & {}^2l^2(x) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & {}^kl^k(x+1) \end{pmatrix}$$

$$- \begin{pmatrix} \sum_j {}_1d_x^{1j} & -{}_1d_x^{12} & \cdots & -{}_1d_x^{1k} \\ -{}_1d_x^{21} & \sum_j {}_1d_x^{2j} & \cdots & -{}_1d_x^{2k} \\ \cdots & \cdots & \cdots & \cdots \\ -{}_1d_x^{k1} & -{}_1d_x^{k2} & \cdots & \sum_j {}_1d_x^{kj} \end{pmatrix}$$

The matrix $\boldsymbol{D}(x)$ is a matrix of increment and decrements. Quantities in the diagonal entries, the cells (i, i), are decrements associated with state i (the sum of all the ${}_1d_x^{ij}$ or exits out of i). By contrast, quantities in off-diagonal columns, the cells (i, j), are increments for state i.

The matrix equation (12.4) preserves all the information contained in equation of type I in expression (12.3). The diagonal elements of matrix $\boldsymbol{l}(x+1)$ will always be smaller than the corresponding elements at age x since the values in the diagonal of matrix $\boldsymbol{D}$ are all decrements. They are like the $l(x)$'s in a simple death process.[6] The number of individuals who are in state i at exactly age $x+1$, which we symbolize as $l^i(x+1)$, must be the sum of the elements in the corresponding column of the matrix $\boldsymbol{l}(x+1)$. Thus, for example, $l^1(x+1)$ is simply the sum of the elements of the first column of $\boldsymbol{l}(x+1)$, namely ${}^1l^1(x+1)+{}^2l^1(x+1)+\cdots+{}^kl^1(x+1)$. However, the similarities with the simple life table stop there. In particular, it is not the case that the sum of decrements for state i over all ages x will add up to $l^i(0)$, as is in fact the case in the single decrement table. Since each of the quantities ${}_1d_x^{ij}$ expresses a frequency of events, the ratio to the number of years lived or total exposure in the interval $(x, x+1)$ will represent a rate. In the notation of the life table we ought to have that

$$ {}_1m_x^{ij} = \frac{{}_1d_x^{ij}}{{}_1L_x^{i}} $$

where ${}_1d_x^{ij}$ is the number of transitions from state i to state j in the age interval, ${}_1L_x^i$ is the total number of person-years lived by those who were in state i during the age interval in a stationary population, and ${}_1m_x^{ij}$ are transition rates for the stationary population.

The observational counterpart to ${}_1m_x^{ij}$ are the quantities ${}_1M_x^{ij}$, or the ratios ${}_1D_x^{ij}/{}_1N_x^i$ of observed transitions from i to j in the age interval to the estimated midperiod population in the age group.

Just as we can arrange the values of ${}_1d_x^{ij}$ in a matrix, so we can create a matrix for the values of the observed transition rates, ${}_1M_x^{ij}$:

$$ \boldsymbol{M}(x) = \begin{pmatrix} \sum_j {}_1M_x^{1j} & -{}_1M_x^{12} & -{}_1M_x^{13} & \cdots & -{}_1M_x^{1k} \\ -{}_1M_x^{21} & \sum_j {}_1M_x^{2j} & -{}_1M_x^{23} & \cdots & -{}_1M_x^{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -{}_1M_x^{k1} & -{}_1M_x^{k2} & -{}_1M_x^{k3} & \cdots & \sum_j {}_1M_x^{kj} \end{pmatrix} \quad \textbf{(12.5)} $$

which is obtained when we divide the quantities in the matrix $\boldsymbol{D}(x)$ by the corresponding values of exposure. This matrix equation is equivalent to equation of type II in expression (12.3).

Let us assume that the functions representing the number of survivors in each state are linear between any two ages, that is, that the values of ${}_1L_x^i$ can be generated as the average of $l^i(x)$ and $l^i(x+1)$. Similarly, we assume that the values ${}_1L_x^{ij} = .5\cdot({}^il^j(x+1))$ are a good representation of the number of person-years lived in the age interval $(x, x+1)$ by those who moved from state i to state j. If so, we can arrange the values of ${}_1L_x^{ij}$ in a matrix and use matrix notation again to write the following:

$$ \boldsymbol{L}(x) = .5\cdot[\boldsymbol{l}(x)+\boldsymbol{l}(x+1)] \quad \textbf{(12.6)} $$

a matrix equivalent of equation of type III in expression (12.3).

Substituting (12.6) into (12.5) we shall obtain the solution for $\boldsymbol{l}(x+1)$ as:

$$ \boldsymbol{l}(x+1) = \boldsymbol{l}(x)\cdot[\boldsymbol{I} - .5\cdot\boldsymbol{M}(x)][\boldsymbol{I} + .5\cdot\boldsymbol{M}(x)]^{-1} \quad \textbf{(12.7)} $$

where $\boldsymbol{I}$ is the identity matrix and the superscript "-1" stands for the inverse of the matrix. This is a formal solution to the simultaneous equations we introduced in expression (12.4). Note that expression (12.7) is the matrix equivalent of the two-state solution where

$$ l(x+1) = l(x)\cdot\frac{1 - .5\cdot{}_1M_x}{1 + .5\cdot{}_1M_x} $$

To calculate $l^i(x+1)$ associated with each state we need to solve for the corresponding values using expression (12.7) for each age group. As in the two-state case, these values are sufficient to calculate all other quantities of interest. The process cannot get started, however, unless we specify a radix, $l^i(0)$, which in our example was set to be $l^j(0) = 1{,}000$ for all j.

We showed that calculations needed to construct increment–decrement life tables are, in principle at least, fairly simple: one needs to invert a matrix for each and every one of the age groups or time intervals considered relevant, and then calculate the quantities of interest from the resulting estimates. These quantities are then assembled in the form of life tables, one for each state. With a few states (less than four) and a handful of age groups, matrix inversion presents few difficulties, and can be done expeditiously with a hand calculator. When the number of states and age groups is larger, however, matrix inversion becomes tedious and can be better handled by a computer. In the last section of this chapter we provide some suggestions regarding software to accomplish these tasks.

12.4.4 Introduction of mortality or other absorbing states

Although in examples such as the one considered above, it is in principle justified to neglect the existence of an absorbing state, this may not always be the case. For example, to model the dynamics of HIV/AIDS or of health conditions at older ages, we will need to explicitly introduce mortality.

The introduction of an absorbing state presents no added difficulties but does require a suitable redesign of matrices and vectors, one that facilitates interpretations and simplifies numerical manipulations. If we were to introduce mortality in the example of children's family life experiences we would need to include an additional state and all associated transitions. The matrices would normally be arranged in such a way as to have death as the last state to be considered (the last row in matrices $\boldsymbol{l}(x)$, $\boldsymbol{l}(x+1)$, $\boldsymbol{M}(x)$). By convention we set the last row of matrices $\boldsymbol{l}(x+1)$, $\boldsymbol{D}(x)$, and $\boldsymbol{M}(x)$ to zero to reflect inactivity in the absorbing state. Aside from these changes in the design of our matrices, no other modifications are required.

12.4.5 Closing the multistate table

As in the case of a simple life table, the calculation of quantities corresponding to the last age group or duration presents some difficulties. In chapter 3 we saw that to close the table we needed to assume that the population was stationary above some high age, ω. This enabled us to set the following equation:

$$ {}_\infty L_\omega = \frac{l(\omega)}{{}_\infty M_\omega} \quad \textbf{(12.8)} $$

to solve for the unknown value of the number of person-years lived.

In the multistate case we proceed in an analogous fashion. The only difference is that we must now account for several states of interest, $i = 1, 2, \ldots, k$. Thus in the example of marriage and union formation and dissolution we need to apply equation (12.8) three times because there are three values ${}_{\infty}L^i_{\omega}$ for which we need to solve. As the reader must have guessed, the operation involved is simply a matrix multiplication:

$$L(\omega) = l(\omega) \cdot [M(\omega)]^{-1} \quad \textbf{(12.9)}$$

where $L(\omega)$ and $l(\omega)$ are diagonal matrices. $M(\omega)$ is a $(k \times k)$ matrix constructed by taking into account only k non-absorbing states.

12.5 The Simplest Case: A Two-state System

In this section we briefly review the explicit solution for the two-state case. We do this because the expressions for the relevant quantities are revealing of the dynamics of the process and of the consequences of some of the underlying assumptions.

Suppose we have a two-state system with no absorbing state. The solution for the conditional probabilities of staying in states 1 and 2 in the age interval x to $x+1$ (${}_1p^{11}_x$ and ${}_1p^{22}_x$) and of moving from state 1 to state 2 and from state 2 to state 1 in the age interval x to $x+1$, (${}_1q^{12}_x = 1 - {}_1p^{11}_x$ and ${}_1q^{21}_x = 1 - {}_1p^{21}_x$), are:

$${}_1p^{11}_x = \frac{1 + .5 \cdot {}_1M^{21}_x - .5 \cdot {}_1M^{12}_x}{1 + .5 \cdot {}_1M^{21}_x + .5 \cdot {}_1M^{12}_x} \qquad {}_1q^{12}_x = \frac{{}_1M^{12}_x}{1 + .5 \cdot {}_1M^{21}_x + .5 \cdot {}_1M^{12}_x}$$

$${}_1p^{22}_x = \frac{1 + .5 \cdot {}_1M^{12}_x - .5 \cdot {}_1M^{21}_x}{1 + .5 \cdot {}_1M^{21}_x + .5 \cdot {}_1M^{12}_x} \qquad {}_1q^{21}_x = \frac{{}_1M^{21}_x}{1 + .5 \cdot {}_1M^{21}_x + .5 \cdot {}_1M^{12}_x}$$

The reader should verify that this solution results from expression (12.7) with the following 2×2 matrix $M(x)$:

$$M(x) = \begin{pmatrix} {}_1M^{12}_x & -{}_1M^{12}_x \\ -{}_1M^{21}_x & {}_1M^{21}_x \end{pmatrix}$$

A comparison of the conditional "survival" probabilities for this case with those obtained in the simple life table is revealing. The expression for the conditional probability of moving from state 1 to state 2 in the simple life table is given by the ratio ${}_1M_x/(1 + .5 \cdot {}_1M_x)$, which is approximately equal to the product of ${}_1M_x \cdot (1 - .5 \cdot {}_1M_x)$. This product expresses directly the implications of the assumption of linearity: it is tantamount to requiring that all events occur at the midpoint of the interval, at which point there should be a fraction of approximately $(1 - .5 \cdot {}_1M_x)$ of the original survivors who will be exposed to an attrition given by ${}_1M_x$.

In the case of a two-state system with two flows the conditional probability of moving from state 1 to state 2 can be interpreted analogously. We first survive individuals in state 1 at age x up to the middle of the interval $(x, x+1)$. We do this by using the quantity $(1 - .5 \cdot {}_1M^{12}_x)$. We then apply the rate ${}_1M^{12}_x$ and the factor $(1 - .5 \cdot {}_1M^{21}_x)$, the latter accounting for the fact that some individuals who move from 1 to 2 will experience a move back to the original state. Thus the probability that an individual in state 1 at exact age x is in state 2 at exact age $x+1$, ${}_1q^{12}_x$, is:

$${}_1M^{12}_x \cdot (1 - .5 \cdot {}_1M^{21}_x) \cdot (1 - .5 \cdot {}_1M^{12}_x)$$

which, when the rates are small, is approximately equal to:

$$\frac{{}_1M^{12}_x}{1 + .5 \cdot {}_1M^{21}_x + .5 \cdot {}_1M^{12}_x}$$

An analogous derivation results in the second equation for ${}_1q^{21}_x$.

12.6 Alternative Solutions: The Case of Constant Rates

The solution expressed by equation equation (12.7) rests on the assumption that the functions ${}^il^j(x)$ are linear in unit intervals. This implies that the underlying risk $\mu^{ij}(a)$ in the unit interval, $(x \le a \le x+1)$, is increasing. In some cases it may be more accurate and convenient to assume that the rates are approximately constant in an interval. This implies that $l(x)$'s are nonlinear (exponential) functions of age. An analogous consequence follows in the multistate case: all the quantities ${}^il^j(x+1)$ become an exponential function of the rates ${}_1M^{ij}_x$. The only caveat here is that we are dealing with an array of functions and that the expressions involve matrices, not scalars. Indeed, the solution for the matrix $l(x+1)$ is now

$$l(x+1) = l(x) \cdot \exp\{-M(x)\} \quad \textbf{(12.10)}$$

where $l(x+1)$, $l(x)$ and $M(x)$ are the same matrices defined before. Expression (12.10) is somewhat meaningless without a definition for the matrix-valued exponential function. Just as in the one-dimensional case, the function $\exp(q)$, where q is any real number, can be expressed as an infinite series of the form $(1 + q + q^2/2! + q^3/3! + \cdots)$, so it is possible to define $\mathbf{exp}(Q)$, where Q is an n × n matrix, as

$$\exp(Q) = I + Q + [Q]^2(1/2!) + [Q]^3(1/3!) + \cdots$$

In most cases the rates will be sufficiently small that only the first or first two elements in the series will be necessary to approximate well the quantity on the left of the expression. If so, the solution for the multistate life table system is even simpler than in the case when $l(x+1)$ was assumed to be linear. This is because no matrix inversion and at most one matrix multiplication is required.

How is one to choose between alternative procedures to estimate the required conditional probabilities of an increment–decrement table? A good answer would be that under very general conditions, the linear method is to be preferred on the grounds of simplicity and ease of calculations. However, it is known that the assumption of linearity leads to a fair amount of inaccuracy when the underlying risks are decreasing rapidly (Schoen, 1988), and that it may even lead to outright impossibly negative values when some or all of the transition rates are very large (Hoem and Funck-Jensen, 1982; Nour and Suchindran, 1984). Thus, the exponential method, or alternatively the so-called "mean duration of transfer method" (Schoen, 1988), are to be preferred on the grounds of consistency.

12.7 Programs for the Calculation of Increment–Decrement Life Tables

There a number of computer programs available for calculation of increment–decrement life tables. In the late seventies, Willekens wrote a quite general program implementing the linear solution, but unfortunately it was not made widely available. The first program to be quite broadly accessible was designed and written by Robert Schoen. The corresponding Fortran code is fully included in his book (Schoen, 1988). The main limitation of this program is that it restricts estimation and calculations to a four-state multistate system.

More recently, Andrei Rogers, a pioneer in the application of increment–decrement tables procedures, made available a DOS-compatible program that performs fairly general calculations using the assumption of linearity (Rogers, 1995b). Since the program runs on any PC with minimum memory requirements, it is an attractive option. Its only limitation is that the number of output functions associated with the estimated life tables is fairly restricted.

Pete Tiemeyer and Glen Ulmer, two former Ph.D. students at the Center for Demography and Ecology, University of Wisconsin, wrote a C++ program that can run on any PC with minimal memory and hard disk space requirements. The program implements the linear solution and can handle any number of states and time intervals (Tiemeyer and Ulmer, 1991). Finally, it outputs a very large number of functions and outcomes. The program with accompanying instructions for installation and implementation is freely available from the authors.

Inevitably each empirical application will demand attention to special conditions, data inputs and outputs. Most of the available software is not general or flexible enough to handle a very broad class of applications or to implement alternative solutions (exponential instead of linear). Thus, in most cases it will be up to the researchers to create their own tool for estimation and calculation of increment–decrement tables. Our suggestion is to use general software packages such as STATA, S-PLUS, or MATLAB that are conducive to mixing preprogrammed routines (such as matrix inversion) with user-defined subroutines (for example those required to estimate conditional life expectancies).

NOTES

1. In this notation ${}_1M_x^{ij}$ corresponds to the transition rate between state i and state j in the age group $x, x+1$. The rates $\lambda^{ij}(x)$ are the continuous version of the ${}_1M_x^{ij}$'s.
2. The reader should remember that the assumption that $l(x)$ is linear in one-year intervals is equivalent to assuming that ${}_1a_x = .5$, and that $\mu(x)$ is a monotonically increasing function of x.
3. It is important to remember that the system of equations in (12.3) rests on the assumption that we know the values of M_x^{ij}. Normally this requires that the observed rates be identical to the ones in the stationary population.
4. The quantities displayed in tables 12.2, 12.3, and 12.4 were obtained using the matrix solution discussed later in this chapter. These estimates are slightly different from those that one would obtain solving the system of 12 equations for each age group. Because of these differences, the empirical relations between estimated quantities (such as ${}_1d_x^{ij}$ and ${}_1M_x^{ij}$) do not exactly correspond to what is implied by the expressions in equation (12.3).
5. Notice that for each j in the table, $e^j(0)$ is NOT equal to the weighted average of the values $\Psi^{ij}(0)$ for $i = 1, 2, 3$.
6. Because $D(x)$ is a matrix of decrements AND increments, there is no compelling justification to write expression (12.4) with a negative sign. We could have just as well have written $l(x) + D(x)$ and changed the sign of the cells of the matrix $D(x)$.

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