LONG-RANGE TRENDS IN ADULT MORTALITY: MODELS AND PROJECTION METHODS*

JOHN BONGAARTS

In the study reported here, I had two objectives: (1) to test a new version of the logistic model for the pattern of change over time in age-specific adult mortality rates and (2) to develop a new method for projecting future trends in adult mortality. A test of the goodness of fit of the logistic model for the force of mortality indicated that its slope parameter is nearly constant over time. This finding suggests a variant of the model that is called the shifting logistic model. A new projection method, based on the shifting mortality model, is proposed and compared with the widely used Lee-Carter procedure.

Over the past two centuries, life expectancy at birth in the industrialized ("developed") world approximately doubled, reaching 79 years for females and 72 years for males in 2000–2005 (United Nations 2002). Much of this rise is attributable to large reductions in infant and child mortality. Mortality among the young is now so low, however, that further declines will have little impact on future trends in life expectancy. Future increases in life expectancy will therefore require additional reductions in adult mortality. In this article, I examine past trends in the age pattern of adult mortality and discuss their implications for long-range mortality projections.

The description of observed age patterns of adult mortality with mathematical models is one of the oldest and most important topics in demography. The number and complexity of mortality models have grown rapidly since Gompertz proposed the first "law of mortality" in 1825. A good model provides a simple but adequate mathematical description of mortality by age and/or time. The objective is to identify fundamental and persistent patterns in the data and summarize them with as few parameters as possible. Models have found many uses, including smoothing of data, construction of model life tables, comparative analyses, testing of theories, and forecasting (Keyfitz 1984; Tabeau, Jeths, and Heathcote 2001).

A concise model description of past mortality trends provides the basis for projections. The theory and practice of forecasting mortality have evolved rapidly in recent decades, and there are many ways to make forecasts (Keyfitz 1991; Lee 1998; Olshansky 1988; Pollard 1987; Tabeau et al. 2001). Projections for the short run typically rely on a simple extrapolation of historical trends in mortality rates, in life expectancy, or in model parameters. However, in projections for periods of more than a few decades, linear extrapolation can lead to implausible results, and expert judgment is then often used to decide which long-range levels or trends are the most probable. For example, experts may identify a target for life expectancy at birth in a future year. This has been the approach

*John Bongaarts, Population Council, 1 Dag Hammarskjold Plaza, New York, NY 10017; E-mail: jbongaarts@popcouncil.org. This research was supported by the Andrew W. Mellon Foundation and the William and Flora Hewlett Foundation. Paul Hewett provided valuable research assistance. An earlier version of this article was presented in the Session on Mathematical Demography at the 2004 annual meeting of the Population Association of America, April 1–3, Boston. Thomas Buettner, Patrick Gerland, Griffith Feeney, Francois Pelletier, John Wilmoth, and two anonymous reviewers provided helpful comments on an early draft.

used by the United Nations, the World Bank, and many national statistical agencies to make long-range population projections.

In the study presented here, I had two objectives: (1) to test a new version of the logistic model to describe the pattern of change over time in age-specific adult mortality rates and (2) to develop a new method for projecting future trends in adult mortality. The first part of the article presents a brief overview of models for the age pattern of adult mortality and a test of the goodness of fit of the logistic model for the force of mortality. This test uses data from the Human Mortality Database for women and men aged 25–109 in 14 populations. The results of this exercise suggest a new version of the logistic model that I call the *shifting logistic model* because the senescent component of adult mortality is assumed to shift to higher ages over time. In the second part of the article, I propose a new projection method that is based on the shifting mortality model. The method is compared with the Lee-Carter procedure, which is one of the most widely used methods for projecting mortality.

MODELS FOR THE FORCE OF MORTALITY

Age Pattern

Mortality rates in a wide range of populations show an approximately exponential rise with age for adults. A simple parametric model proposed by Gompertz (1825) summarizes this pattern:

$$\mu(x) = \alpha e^{\beta x}, \tag{1}$$

where $\mu(x)$ denotes the force of mortality at age x. The two parameters α and β are positive; α varies with the level of mortality, and β measures the rate of increase in mortality with age.

For many purposes, the Gompertz model provides a satisfactory fit to adult mortality rates. However, a close inspection of the difference between model estimates and observed death rates often reveals systematic underestimation of actual mortality at youngest adult ages (younger than 40) and overestimation at the oldest ages (over 80). The deviation at lower ages was addressed by Makeham (1860) with the addition of a constant to the Gompertz model:

$$\mu(x) = \alpha e^{\beta x} + \gamma. \tag{2}$$

The new parameter γ is usually referred to as background mortality, which is the same for all ages. A detailed analysis of Eq. (2) was provided by Gavrilov and Gavrilova (1991).

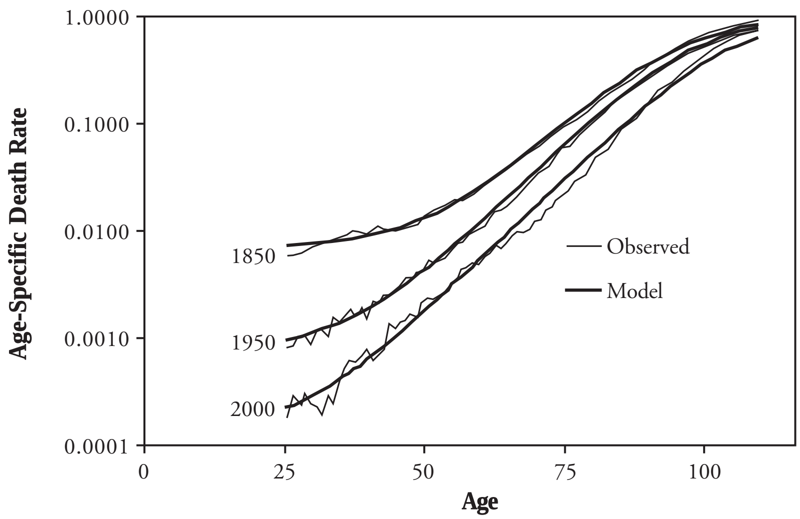
The Makeham model represents a clear improvement over the Gompertz model at younger ages, but it still overestimates mortality at the oldest ages. This deviation can be addressed in a number of ways, most simply by the following logistic model (Thatcher 1999; Thatcher, Kannisto, and Vaupel 1998):

$$\mu(x) = \frac{\alpha e^{\beta x}}{1+\alpha e^{\beta x}} + \gamma. \tag{3}$$

At lower adult ages the force of mortality estimated with Eqs. (3) and (2) are similar because the denominator of the first term in Eq. (3) is close to 1.0. At the oldest ages, however, the two models diverge as Eq. (3) levels off at $1+\gamma$, while Eq. (2) has no limit.

More-complex logistic models with additional parameters have also been proposed (Beard 1971; Horiuchi and Wilmoth 1998; Perks 1932; Thatcher et al. 1998). On the basis of a detailed comparison of different models, Thatcher (1999) and Thatcher et al. (1998) recommend Eq. (3) because it provides an excellent fit to mortality rates over the entire adult age range with relatively few parameters.

Figure 1. Age-Specific Death Rates for Swedish Women, Observed and Estimated With Logistic Model



The good fit of the logistic model in Eq. (3) is demonstrated in Figure 1, which presents observed and estimated values of the force of mortality $\mu(x)$ for Swedish women aged 25–109 in 1850, 1950, and 2000. The proportion of the variance explained by the model equals 0.9994 in 1850, 0.9996 in 1950, and 0.9985 in 2000.

To confirm these results for other populations, the model given by Eq. (3) was fitted to annual mortality data from 1950 to 2000 for 14 countries, separately for men and women. All countries in the Human Mortality Database outside Eastern European were included. Table 1 presents averages of annual estimates for the three parameters in the logistic model (α, β, and γ) for women and men aged 25–109 in each of the 14 countries for all available years 1950–2000.[1] These results are discussed in detail later, but for now it should be noted that the model fits well in all these countries (see the next-to-last column in Table 1). The fit is about the same for women (R^2 averages 0.9993 for the 14 countries) as for men (R^2 averages 0.9996).

Although the simple logistic model is well suited for my present purposes, its fit is not perfect. An examination of differences between observed and fitted values reveals small systematic overestimation of mortality between ages 60 and 80, as well as some underestimation at the highest ages among women in a number of countries. This pattern is consistent with the findings of Himes, Preston, and Condran (1994).

In the following analysis of trends in adult mortality, it is useful to distinguish between senescent mortality, which rises with age, and background mortality, which does

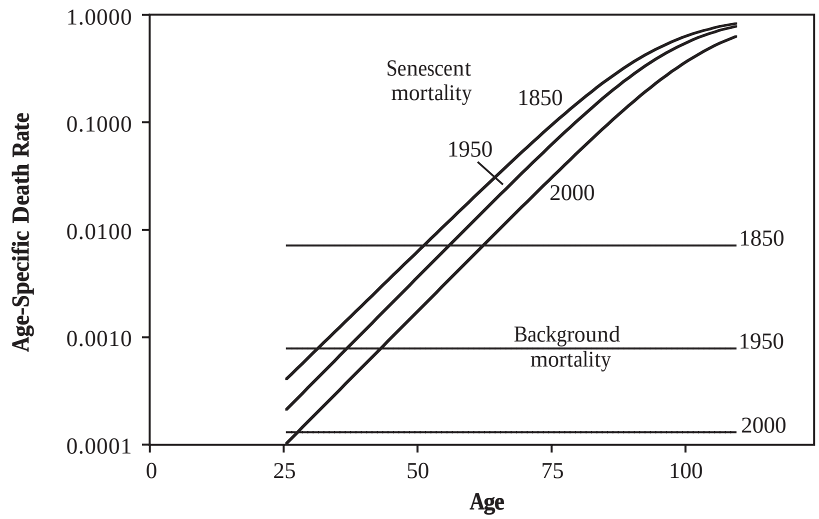
1. Data are available for most years from 1950 to 2000 in the 14 countries, but in several cases, data for the last year(s) in the late 1990s or the early 1950s are missing. For details, see http://www.mortality.org. The nonlinear least-squares routine in STATA was used to obtain estimates of the parameters in the logistic model.

Table 1. Parameters of the Logistic Model for Adult Mortality Fitted to Observed Age-Specific Death Rates for Ages 25–109, Average of Annual Estimates for All Available Years From 1950 to 2000 in 14 Countries

	$\alpha(t) \times 10^5$ (level)		$\beta(t)$ (slope)		$\gamma(t)$ (background)		R^2	R^2
	Average	Coefficient of Variation	Average	Coefficient of Variation	Average	Coefficient of Variation	Variable $\beta(t)$, Average	Constant β, Average
Females								
Austria	0.87	0.310	0.117	0.016	0.00052	0.512	0.9991	0.9991
Canada	1.55	0.292	0.106	0.019	0.00035	0.389	0.9996	0.9996
Denmark	1.52	0.203	0.108	0.042	0.00029	0.658	0.9988	0.9987
England	1.42	0.184	0.109	0.016	0.00027	0.729	0.9997	0.9997
Finland	0.75	0.349	0.119	0.019	0.00053	0.633	0.9991	0.9991
France	0.85	0.443	0.115	0.027	0.00068	0.341	0.9992	0.9991
Italy	0.73	0.346	0.118	0.020	0.00052	0.556	0.9996	0.9996
Japan	0.76	0.628	0.118	0.033	0.00093	0.969	0.9996	0.9995
Netherlands	0.76	0.181	0.116	0.016	0.00035	0.304	0.9994	0.9993
Norway	0.65	0.189	0.117	0.016	0.00032	0.465	0.9992	0.9992
Sweden	0.69	0.290	0.117	0.019	0.00038	0.330	0.9992	0.9992
Switzerland	0.62	0.551	0.120	0.031	0.00047	0.301	0.9991	0.9991
United States	2.18	0.253	0.101	0.018	0.00042	0.183	0.9996	0.9996
West Germany	0.85	0.228	0.116	0.011	0.00046	0.346	0.9994	0.9994
Average	1.01	0.318	0.114	0.022	0.00046	0.480	0.9993	0.9993
Males								
Austria	2.98	0.215	0.106	0.018	0.00097	0.267	0.9995	0.9994
Canada	3.97	0.333	0.100	0.039	0.00066	0.180	0.9996	0.9995
Denmark	2.66	0.278	0.106	0.039	0.00057	0.296	0.9994	0.9993
England	2.82	0.272	0.107	0.020	0.00032	0.482	0.9995	0.9995
Finland	5.77	0.351	0.099	0.035	0.00088	0.473	0.9994	0.9993
France	4.20	0.249	0.101	0.019	0.00098	0.242	0.9995	0.9995
Italy	2.54	0.332	0.107	0.032	0.00076	0.431	0.9996	0.9996
Japan	2.23	0.366	0.108	0.017	0.00104	0.809	0.9998	0.9998
Netherlands	1.99	0.318	0.109	0.036	0.00042	0.421	0.9996	0.9995
Norway	1.96	0.330	0.109	0.039	0.00067	0.334	0.9996	0.9995
Sweden	1.48	0.299	0.112	0.030	0.00073	0.207	0.9997	0.9996
Switzerland	1.80	0.408	0.111	0.035	0.00090	0.236	0.9994	0.9994
United States	6.36	0.412	0.094	0.041	0.00087	0.348	0.9998	0.9996
West Germany	2.92	0.173	0.105	0.017	0.00070	0.297	0.9998	0.9998
Average	3.12	0.310	0.105	0.030	0.00075	0.359	0.9996	0.9995

Source: Estimated from data in the Human Mortality Database.

Figure 2. Model Estimates of the Senescent and Background Death Rates of Swedish Women, by Age



not vary with age (Gavrilov and Gavrilova 1991; Horiuchi and Wilmoth 1998; Makeham 1860). The sum of these two components equals the force of mortality:

$$\mu(x,t) = \mu_s(x,t) + \mu_b(t), \tag{4}$$

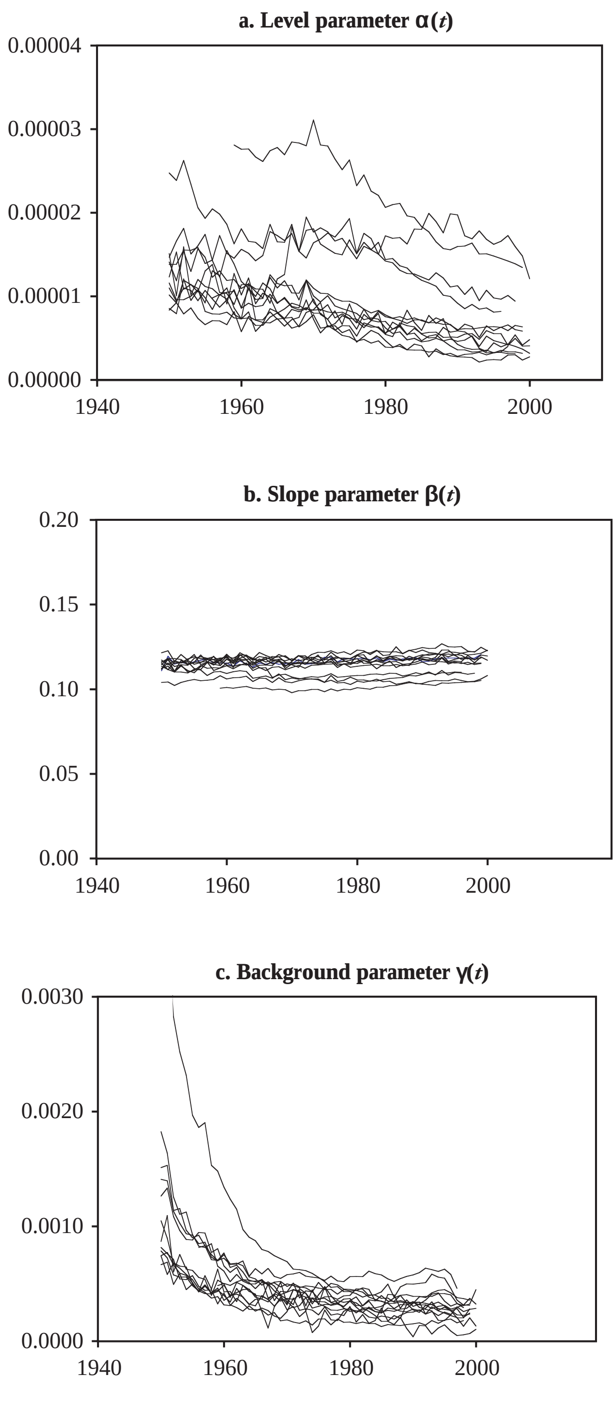
where $\mu_s(x,t)$ is the senescent force of mortality and $\mu_b(t)$ is the background force of mortality. For the logistic model, the first term on the right-hand side of Eq. (3) equals the senescent force of mortality $\mu_s(x,t)$, and the background parameter $\gamma(t)$ equals the background force of mortality $\mu_b(t)$.

Figure 2 plots model estimates of these two components for Swedish women in 1850, 1950, and 2000. The senescent component rises linearly from age 25 to about age 75 because in this age range, the denominator of the senescent component of Eq. (3) is close to 1.0, and the remaining exponential term in the numerator becomes a straight line when plotted on a logarithmic scale, as is the case in Figure 2. At ages older than about 75, the rate of increase in the force of mortality with age declines in the logistic model, and at very high ages, the senescent force of mortality $\mu_s(x,t)$ approaches 1.0. The age-invariant background component (plotted as horizontal lines in Figure 2) has declined sharply over time, from 0.0071 in 1850 to 0.00078 in 1950 and to 0.00013 in 2000. The senescent and background components in Figure 2 add up in each year to the overall model estimate of the force of mortality plotted in Figure 1. At high ages, background mortality is small compared with senescent mortality, and it may be ignored for many analytic purposes, especially in contemporary countries with high life expectancy.

Trends Over Time

Trends in adult mortality can be summarized with time series of the three parameters of the logistic model, $\alpha(t)$, $\beta(t)$, and $\gamma(t)$. Panels a–c of Figure 3 present estimated trends in these parameters for each of the 14 countries from 1950 to 2000. Several conclusions can

Figure 3. Estimates of Level Parameter α, Slope Parameter β, and Background Parameter γ in the Logistic Model for 14 Countries: Women, 1950–2000



be drawn from these results. There is considerable variation among countries in the level parameter $\alpha(t)$, but the trend in this parameter is typically downward. The same is true for background mortality $\gamma(t)$ (see Panel c of Figure 3), but there is less variation among countries in $\gamma(t)$ than in $\alpha(t)$. In addition, declines in background mortality are confined mostly to the period 1950 to 1975. After 1975 there appears to be little systematic trend in $\gamma(t)$ in many of these countries, suggesting that background mortality has reached a low-level plateau.

The most interesting finding in Panel b is that the values of the slope parameter $\beta(t)$ are nearly constant for each population. Similar patterns are observed for men (data not shown). This finding confirms earlier observations by Gavrilov and Gavrilova (1991) and Thatcher (1999). In fact, the near-constancy of the slope parameter extends further into the past. For example, for Swedish women $\beta(t)$ averaged 0.112 for 1850–1900, 0.119 for 1900–1950, and 0.117 for 1950–2000.

The conclusion about the lack of variation with time in the slope parameter $\beta(t)$ is confirmed in Table 1, which presents averages of annual estimates of the coefficients of variation in the parameters of the logistic model $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ for women and men in 14 countries for the period 1950–2000. The coefficient of variation of the slope parameter $\beta(t)$ is small, averaging just 2.2% for women and 3.0% for men. In contrast, the coefficients of variation for the level and background parameters, $\alpha(t)$ and $\gamma(t)$, are at least an order of magnitude larger for both men and women. Clearly, the level and background parameters are much more variable than the slope parameter.

Shifting Logistic Model

The finding that slope parameter $\beta(t)$ is nearly constant suggests a variant of the logistic model in which this parameter is assumed to be fixed over time for a population. The senescent component of the standard model in Eq. (3) then simplifies to

$$\mu_s(x,t) = \frac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta x}}, \tag{5}$$

and the total force of mortality is given by

$$\mu(x,t) = \frac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta x}} + \gamma(t). \tag{6}$$

In this formula, $\alpha(t)$ and $\gamma(t)$ are the only time-varying parameters. The value of the slope parameter β can differ among populations and may take different values for men and women, but it is constant with respect to time.

A change in the senescent force of mortality from $\mu_s(x,t_0)$ at time t_0 to $\mu_s(x,t)$ at time t is conventionally interpreted as a rise or decline in mortality rates. Eq. (5) offers an alternative and unconventional description of changes in the force of mortality. Instead of interpreting mortality as rising or falling, the schedule of the force of senescent mortality can be viewed as shifting to higher or lower ages over time. This interpretation is possible because Eq. (5) has an interesting and useful property: the age pattern of the senescent force of mortality $\mu_s(x,t)$ at time t is the same as at an earlier time t_0, except that the function has shifted to higher (lower) ages as senescent mortality falls (rises). The senescent force of mortality at age x in year t is identical to the value in an earlier year t_0 at age $x - S(t)$ except around age 0. As a result, Eq. (5) can be written as

$$\mu_s(x,t) = \frac{\alpha(t_0)e^{\beta(x-S(t))}}{1+\alpha(t_0)e^{\beta(x-S(t))}}, \tag{7}$$

where $S(t)$ equals the amount of the shift in years up or down the age axis between t_0 and t (Eq. (7) holds for $x > S(t)$, and $\mu_s(x,t) = 0$ for $x < S(t)$). As shown in Appendix A, the conventional up-down and the alternative shifting interpretations are formally equivalent for the logistic model with

$$S(t) = -\frac{\ln(\alpha(t)/\alpha(t_0))}{\beta}. \tag{8}$$

That is, a change in the senescent force of mortality between t_0 and t can be described with Eq. (5) as a change in the level parameter $\alpha(t_0)$ to $\alpha(t)$ or equivalently with Eq. (7) as a shift by $S(t)$ years.

The idea of a shifting mortality schedule can be clarified further by introducing the *senescent life expectancy at birth*, denoted as $e_s(t)$ and defined as

$$e_s(t) = \int_0^\infty \exp\left\{-\int_0^a \mu_s(x,t)dx\right\}da. \tag{9}$$

It equals the average age at death of a newborn, subject to the senescent force of mortality $\mu_s(x,t)$, assuming no background mortality and no nonsenescent mortality at younger ages. The shift to higher or lower ages in the force of the senescent mortality function between t_0 and t is closely approximated by the change in senescent life expectancy between t_0 and t:

$$S(t) \approx e_s(t) - e_s(t_0) \tag{10}$$

because $\mu_s(x,t)$ is very small around age 0.

The pattern of the force of senescent mortality $\mu_s(x,t)$ given by Eqs. (5) and (7) is referred to as the *shifting logistic model*. It is a member of a more general class of models for which the *shifting assumption* holds with

$$\mu_s(x,t) = \mu_s(x - S(t),t_0). \tag{11}$$

The shifting logistic model always implies Eq. (11), but the shifting assumption given by Eq. (11) may hold even when $\mu_s(x,t)$ does not follow a logistic pattern (discussed later). It should be emphasized that, in general, the shifting property applies only to senescent mortality, not to all adult mortality.

The shifting is evident in Figure 2, where the lines for senescent mortality in 1850, 1950, and 2000 have similar shapes, with the schedules for later years moved to higher ages compared with earlier years. The shift equaled four years between 1850 and 1950 and seven years between 1950 and 2000. A shifting pattern for mortality change was proposed earlier by Kannisto (1996), and some of its implications were examined by Bongaarts and Feeney (2002, 2003).

The shifting logistic model describes changes over time in the age pattern of senescent mortality with only one time-varying parameter (either the level $\alpha(t)$ or the shift $S(t)$). This advantage is offset by some loss in the goodness of fit. However, the proportion of variance explained by the shifting model with constant slope parameter β is still an impressive 0.9993 for women and 0.9995 for men (average of 14 populations and all years from 1950 to 2000). The last column of Table 1 presents the R^2 values for men and women in each of the 14 countries, with the slope β held constant at its average for 1950–2000. These results are only slightly smaller than the R^2 for the logistic model with a variable $\beta(t)$ presented in the next-to-last column in Table 1.

These results indicate that the shifting logistic model provides a good general description of age patterns of adult mortality in many countries for the past half century. The implications of this finding for mortality projections are discussed next (see Appendix B for an analysis of the implications of the model for rates of change in mortality).

PROJECTING ADULT MORTALITY RATES

The models for the force of mortality discussed in the preceding section are now applied to gain insights into projection methods. After a brief description of the Lee-Carter method, a new forecasting approach is proposed.

The Lee-Carter Method

Lee and Carter (1992) described a new statistical method for modeling and forecasting mortality by age that has been adopted widely. For example, the U.S. Census Bureau uses the Lee-Carter forecast as a benchmark for its long-run forecast of life expectancy (Hollman, Mulder, and Kallan 2000), and a Social Security Technical Advisory Panel recommended the adoption of the method (Lee and Miller 2001; Technical Panel on Assumptions and Methods 1999). Projections of mortality for the G7 countries by Tuljapurkar, Li, and Boe (2000) also used this method. On the basis of the recommendations of an expert group, the United Nations Population Division has prepared its long-range projections to 2300 for all countries in the world with a variant of the Lee-Carter model (United Nations 2003). Recent discussions of the model and its applications can be found in Booth, Maindonald, and Smith (2002), Carter and Prskawetz (2001), Lee (1998, 2000), Lee and Miller (2001), and Tabeau et al. (2001).

The Lee-Carter method is based on the following mortality model:

$$\ln[m(x,t)] = a(x) + b(x)\kappa(t) + \varepsilon(x,t), \tag{12}$$

where $m(x,t)$ is the central death rate at age x and time t, $\kappa(t)$ is the index of level of mortality, $a(x)$ are age-specific constants describing the general pattern of mortality by age, $b(x)$ are age-specific constants for the relative speed of mortality change, and $\varepsilon(x,t)$ is the residual. This model provides a good fit to past age-specific mortality rates in the United States, explaining 93% of the within-age-group variance between 1900 and 1987 (Lee and Carter 1992).

Eq. (12) provides the basis for making mortality projections. A projection requires only the extrapolation of the index $\kappa(t)$ because $a(x)$ and $b(x)$ are estimated from past data and are held constant for the duration of the projection. An ARIMA time-series model is usually used for the index $\kappa(t)$, and Lee and Carter (1992) and other analysts have assumed a random walk with a drift that describes past trends in $\kappa(t)$ well. The implication of assuming a linear trend in $\kappa(t)$ to continue into the future is that mortality rates at all ages follow an exponential decline. That is, the projected proportional rate of mortality decline $\rho(x,t)$ in a future year t varies by age, but it is assumed to remain equal at each age to the rate observed in the past:

$$\rho(x,t) = \rho_h(x), \tag{13}$$

where $\rho_h(x,)$ is the observed rate of decline in the death rate at age x over some historical period h that ends in the base year of the projection. To ensure robust results, Lee and Carter recommended that estimates of past rates of decline $\rho_h(x)$ be based on historical data for periods of several decades.

The model has several attractive features: a relatively simple demographic model captures the main trends in patterns of past mortality change, forecasting is based on persistent long-term trends and involves no subjective judgment, and the application of statistical time-series methods provides probabilistic confidence intervals for the forecast (Booth et al. 2002; Lee and Miller 2001). In addition, tests in several populations have indicated that the projections made with this method are accurate over short time horizons (Lee and Miller 2001). The Lee-Carter method, however, also has a limitation that becomes increasingly significant as the projection duration rises. The central assumption that the rate of decline in mortality at each age remains invariant over time have been violated in several countries in recent decades. Instead of being constant, rates of improvement in mortality have tended to decline over time at younger ages, while they have risen at older ages (Booth et al. 2002; Carter and Prskawetz 2001; Lee 2000; Lee and Miller 2001). Appendix B confirms that the age pattern of the rate of mortality decline $\rho(x,t)$ has varied over time and examines the factors underlying this variation. This finding implies that rates of improvement are likely to continue to change in the future. By not allowing such change, the Lee-Carter method may produce implausible results in projections over many decades. Some investigators have attempted to address this limitation by adding complexity and additional parameters to the Lee-Carter model (Booth et al. 2002; Carter and Prskawetz 2001). The alternative approach proposed next provides a simpler solution.

A New Projection Procedure

The shifting logistic model suggests several ways to project future age-specific rates of adult mortality. The simplest approach consists of fitting the logistic model with a fixed-slope parameter to past data, followed by extrapolation of the model parameters. The preparation of such a basic projection consists of the following four steps:

1. Fit the three-parameter logistic model in Eq. (3) to mortality schedules for a selected period before the base year of the projection. This fitting exercise produces time series for each of the three parameters in the logistic model, $\alpha(t)$, $\beta(t)$, and $\gamma(t)$. As I discussed earlier, Figure 3 illustrates this step for women in 14 countries from 1950 to 2000. (These estimates were obtained with the nonlinear least-squares routine in STATA.)

2. Fix the value of the slope parameter β at its average value and fit the two-parameter model in Eq. (6) to the same data. The resulting time series of level parameter $\alpha(t)$ and background parameter $\gamma(t)$ differ slightly from those obtained in Step 1. Figure 4 presents these new estimates for the same 14 countries from 1950 to 2000 (note the logarithmic scale).

3. Extrapolate the level parameter $\alpha(t)$ and background parameter $\gamma(t)$ obtained in Step 2 for the desired duration of the projection.[2]

4. Construct future adult mortality schedules as logistic curves using Eq. (6), based on the extrapolated values of parameters $\alpha(t)$ and $\gamma(t)$.

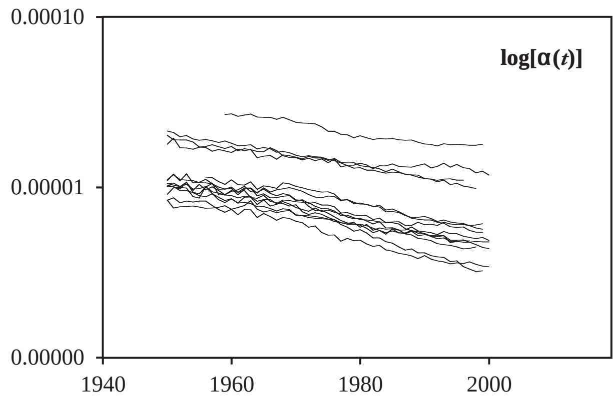
This four-step projection procedure is straightforward and relatively easy to apply, and it should give satisfactory results in populations in which the logistic model in Eq. (6) fits well. However, in some populations, the differences between the observed and fitted mortality rates may be significant and systematic at some ages. In such cases, one of the following variants of the foregoing basic projection procedure will be preferable:

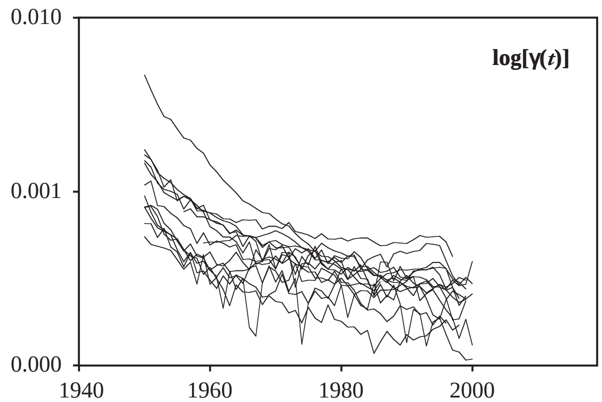
Variant 1. Although the standard logistic model in Eq. (5) fits well, more-complex logistic models provide an even better fit. As Thatcher (1999) noted, a simple four-parameter logistic model can be obtained by multiplying the numerator of the senescent force of mortality $\mu_s(x,t)$ in Eq. (5) by an additional parameter λ. In the three-parameter model used in this study, λ is assumed to be equal to 1. Allowing λ to deviate from 1 should provide a better general description of age patterns of mortality, particularly at the highest ages. In projections λ may be held constant, as is the case now for slope parameter β. The implementation of this variant follows the previous four steps, but in Step 1, four parameters—$\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\lambda(t)$—are estimated, and in Step 2, $\beta(t)$ and $\lambda(t)$ are held constant while $\alpha(t)$ and $\gamma(t)$ are reestimated.

Variant 2. An approach that is still more flexible is not to rely on a logistic model in Step 4 and to assume only that the shifting assumption holds in the future. That is, instead of constructing future adult mortality schedules as logistic curves, senescent mortality is

2. To avoid the influence of random fluctuations in past estimates of $\alpha(t)$ and $\gamma(t)$ on the projection, it is usually desirable to smooth these time series by taking a three- or five-year moving average before extrapolating.

Figure 4. Estimates of Level Parameter α and Background Parameter γ in the Shifting Logistic Model (Constant β) for Women in 14 Countries





projected as a shifted version of the observed schedule in the base year, using Eq. (11). In this approach, the age pattern of senescent mortality observed in the base year is shifted to higher ages in future years while maintaining its original shape.[3] The amount of the shift $S(t)$ for each future year is estimated from the projected trend in level parameter $\alpha(t)$ with Eq. (8). Background mortality is projected separately on the basis of the extrapolation of background parameter $\gamma(t)$.

The choice of whether to use the basic method or one of these two variants should be informed by how well the simple three-parameter logistic model fits recent observed mortality rates. If the fit is extremely good and only random deviations are present, then the basic method may be adequate. If the fit is good at all ages except among the oldest-old,

3. The baseline estimate of $\mu_s(x,t_0)$ is obtained by subtracting the estimated background mortality $\gamma(t_0)$ from the observed force of mortality $\mu(x,t_0)$.

then Variant 1, with the four-parameter logistic, is a better choice. Finally, if systematic deviations between model and observed rates are present, then Variant 2 should give the best results. This variant is more complex to implement, but it has clear advantages: there are no discontinuities in trends in age-specific mortality rates at the beginning of the projection, and the country-specific features of the mortality schedule that are not captured by the logistic are preserved in the projection.[4]

A brief comment on the method of extrapolation of the level and background parameters is in order. As shown in the top graph in Figure 4, the values of the level parameter, as measured by $\log[\alpha(t)]$, have declined at a nearly constant pace over the past half century in the 14 countries. Extrapolation of $\log[\alpha(t)]$ can therefore rely on the same time-series model used by Lee-Carter to project their index $\kappa(t)$: a random walk with a drift. The near-linear trends in $\log[\alpha(t)]$ that are observed in the top graph of Figure 4 suggest that it is reasonable to assume that this trend will continue for a few more decades, but it is not clear whether this will be the case in very long-range projections. It is interesting to note that, according to Eqs. (8) and (10), linear extrapolation of $\log[\alpha(t)]$ yields a linear extrapolation of senescent life expectancy, $e_s(t)$.

The same extrapolation approach can be used to project values of the logarithm of background mortality. However, it is evident from the bottom graph in Figure 4 that the pace of decline in $\log[\gamma(t)]$ is not as steady as in $\log[\alpha(t)]$. Alternative nonlinear methods may therefore be preferable for extrapolating background mortality, but these methods are not examined further here. Values of background mortality have reached such low levels in contemporary developed countries that a small error is of little consequence in projections.

Since this article focuses on adult mortality, no attempt is made to propose alternative ways to project child and young adult mortality. Moreover, the shifting model does not apply to mortality patterns at ages younger than 25. No improvement over conventional methods can therefore be suggested. Further details on projection methods for the youngest age groups can be found in Pollard (1987), Tabeau et al. (2001), and United Nations (2002).

Illustrative Applications

A detailed evaluation of the new projection methods and comparison with the Lee-Carter approach are beyond the scope of this article, but Figure 5 presents an illustrative application for adult mortality of Swedish women. The initial year for the projection is 1975, and the projection from 1975 to 2000 is based on an extrapolation of estimated parameters from 1950 to 1975. Figure 5 presents three mortality schedules for 2000: the observed one and two projections, obtained with the new method and with the Lee-Carter method.[5] Both methods performed well, and their projected age-specific mortality rates are similar to the observed rates in 2000. The observed life expectancy at age 25 rose from 49.9 years in 1950 to 54.12 years in 1975 to 57.5 years in 2000. The projected life expectancies at age 25 in 2000 were 57.7 years for the new method and 58.0 years for the Lee-Carter method. In this application, the differences between the two methods are minor.

Figure 6 presents another illustrative application in which the initial year for the projection is 2000, and the projection from 2000 to 2100 is based on an extrapolation of estimated parameters from 1950 to 2000. In this long-range projection, the two methods

4. The new projection method can also be applied in populations for which mortality data are available only for a single year or period. This is the case for many developing countries, where mortality data are often limited. The available information for one period provides the baseline estimates of levels of background and senescent mortality, but in the absence of past data, analysts will have to make assumptions about future trends in the parameters $\alpha(t)$ and $\gamma(t)$.

5. Annual estimates of parameters and age-specific death rates from 1950 to 1975 that are used in these projections were smoothed by taking a five-year moving average.

Figure 5. Comparison of Alternative Projections of Death Rates in 2000 (1975 Base Year): Swedish Women

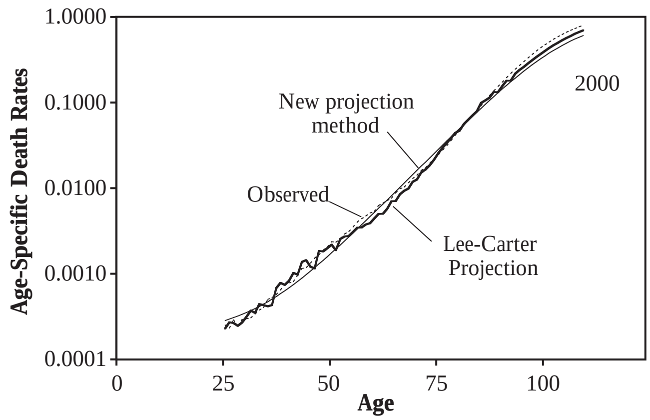
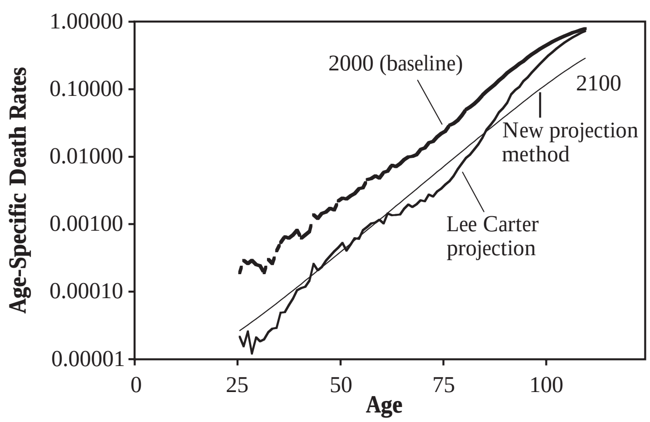


Figure 6. Comparison of Alternative Projections of Death Rates to 2100: Swedish Women



produce different age patterns in 2100. Compared with the new method, the Lee-Carter projection expects little improvement in mortality at the highest ages and large improvements in the 60–80 age group. It is not obvious which projection is preferable. However, a simple theoretical argument supports the view that the new approach gives more-robust long-range projections for the age pattern of mortality than does the Lee-Carter method. The Lee-Carter method is equivalent to extrapolating past trends in mortality rates for each age group at its own exponential rate (Lee and Miller 2001; McNown 1992). This method is potentially problematic because any differences between the $b(x)$ values of successive age groups will eventually cause differences between the projected mortality rates of these age groups either to become very large or to turn negative. In either case, the Lee-Carter method may forecast implausible age patterns in the very long run. In contrast, the new method ensures that the age structure of senescent mortality remains plausible, regardless of the duration of the projection.

CONCLUSION

Past age patterns in the force of mortality among adults are well described with a simple logistic model in which the slope parameter is assumed to be constant over time within each population. The model includes separate components for background and senescent adult mortality, each of which is summarized with one time-varying parameter. Despite its simplicity, this model captures the main features of complex changes over time in age-specific mortality rates among adults.

The constancy of the slope parameter in this model implies that the senescent component of the force of mortality shifts to higher or lower ages as mortality conditions improve or deteriorate for adults. This shifting model introduces an alternative way of thinking about changes in mortality. The conventional view is that a change in senescent mortality implies increases or decreases in age-specific mortality rates. The proposed new view considers a change in senescent mortality rates to be the result of delays in the timing of death. This alternative perspective is captured in the shifting logistic model, which provides a parsimonious description of past trends in senescent mortality.

The shifting mortality model also provides the basis for a new method for making projections of age-specific mortality that has certain advantages over existing procedures. In particular, the method addresses a key weakness in the Lee-Carter method (i.e., the assumption that the age-specific rate of decline in mortality remains constant over time). Further research is needed to establish whether and under what conditions the proposed new method produces more-accurate projections than do existing methods.

APPENDIX A: RELATIONSHIP BETWEEN LEVEL PARAMETER $\alpha(t)$ AND THE SHIFT IN THE SENESCENT FORCE OF MORTALITY

A decline in the value of the level parameter from $\alpha(t_0)$ at time t_0 to $\alpha(t)$ at time t implies a decline in senescent mortality from $\mu_s(x,t_0)$ to $\mu_s(x,t)$ as estimated from Eq. (5).

Let the ratio of $\alpha(t)$ to $\alpha(t_0)$ be denoted $p(t)$, with

$$p(t) = \frac{\alpha(t)}{\alpha(t_0)}. \tag{A1}$$

Substitution of Eq. (A1) in Eq. (5) gives

$$\mu_s(x,t) = \frac{p(t)\alpha(t_0)e^{\beta x}}{1 + p(t)\alpha(t_0)e^{\beta x}}$$

$$= \frac{\alpha(t_0)e^{\beta(x+\ln(p(t))/\beta)}}{1+\alpha(t_0)e^{\beta(x+\ln(p(t))/\beta)}}. \tag{A2}$$

Define

$$S(t) = -\frac{\ln(p(t))}{\beta} = -\frac{\ln(\alpha(t)/\alpha(t_0))}{\beta}. \tag{A3}$$

Substitution of Eq. (A3) in Eq. (A2) gives

$$\mu_s(x,t) = \frac{\alpha(t_0)e^{\beta(x-S(t))}}{1+\alpha(t_0)e^{\beta(x-S(t))}}$$
$$= \mu_s(x - S(t), t_0). \qquad \text{(A4)}$$

A decline in α between t_0 and t is equivalent to a shift of $S(t)$ years in the schedule of the force of mortality, with $S(t)$ given by Eq. (A3).

APPENDIX B. MODELS FOR THE RATE OF CHANGE IN THE FORCE OF MORTALITY

Past studies of mortality trends have yielded important insights by examining the rate of change in the force of mortality with respect to age or time (Horiuchi and Coale 1990; Horiuchi and Wilmoth 1998; Keyfitz 1977; Vaupel 1986; Vaupel and Romo 2003). The shifting model provides useful insights into the factors that determine trends in the rate of change in the force of mortality.

Rate of Change by Age

The relative derivative of the force of mortality $\mu(x,t)$ with respect to age is defined as

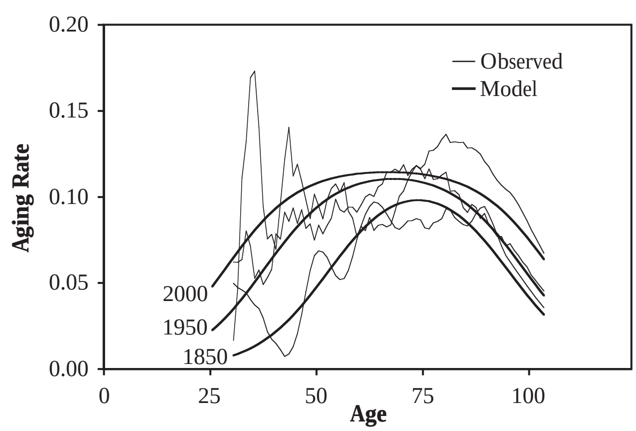
$$k(x,t) = \frac{1}{\mu(x,t)}\frac{\partial\mu(x,t)}{\partial x} \qquad \text{(B1)}$$

and is referred to as the age-specific rate of mortality change with age (Horiuchi and Coale 1990) or the life-table aging rate (Horiuchi and Wilmoth 1998).

Figure B1 plots observed and model estimated values of $k(x,t)$ for Swedish women in 1850, 1950, and 2000. The pattern is bell shaped and varies over time. It is also somewhat different for men than for women (data not shown).

To interpret these changes with age and over time, it is useful to decompose $k(x,t)$ into two additive factors representing the senescent component $k_s(x,t)$ and the

Appendix Figure B1. Life-Table Aging Rate, Observed and Estimated With the Shifting Logistic Model: Swedish Women



background component $k_b(x,t)$ (see Horiuchi and Wilmoth 1998 for a slightly different decomposition):

$$k(x,t) = k_s(x,t) + k_b(x,t). \tag{B2}$$

The senescent component $k_s(x,t)$ is defined as the aging rate that would be observed in the absence of background mortality, and $k_b(x,t)$ equals the difference between $k(x,t)$ and $k_s(x,t)$.

As shown in Appendix C, for the shifting logistic model,

$$k_s(x,t) = \frac{\beta}{1+\alpha(t)e^{\beta x}} = \beta[1-\mu_s(x,t)] \tag{B3}$$

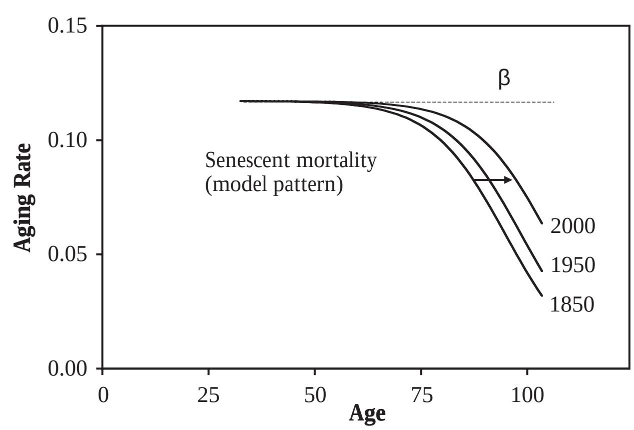
$$k_b(x,t) = \frac{-\beta}{1+[1+1/\gamma(t)]\alpha(t)e^{\beta x}}. \tag{B4}$$

Figure B2 plots the model senescent component $k_s(x,t)$ for Swedish women in 1850, 1950, and 2000. At the youngest ages, $k_s(x,t)$ is approximately constant and equal to β because $\mu_s(x,t) \ll 1$. With advancing age, $k_s(x,t)$ declines and reaches 0 at very high ages. The schedules for $k_s(x,t)$ and $\mu_s(x,t)$ shift together to higher (lower) ages as senescent life expectancy rises (falls).

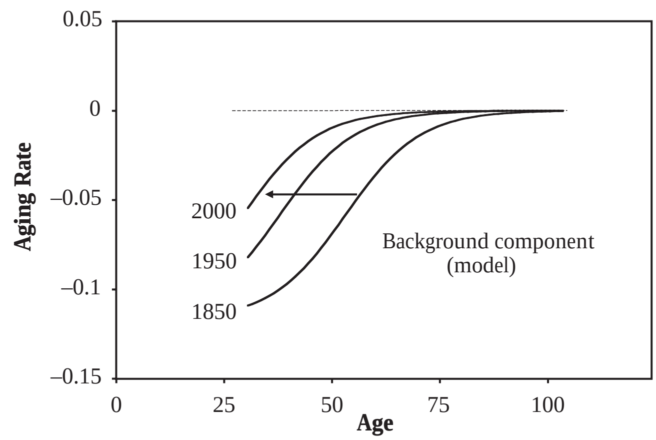
Figure B3 plots model estimates of the background component $k_b(x,t)$ for Swedish women in 1850, 1950, and 2000. The value of $k_b(x,t)$ is negative and rises from $-\beta$ at very young ages to 0 at the oldest ages. An interesting property of the $k_b(x,t)$ schedule is that it shifts to higher or lower ages. But, in general, this shifting occurs at a different rate from the shifting in $\mu_s(x,t)$ and $k_s(x,t)$. In most countries, $k_b(x,t)$ either moves more slowly to the right than does $k_s(x,t)$ (when $\gamma(t)$ declines but less rapidly than $\alpha(t)$) or shifts to the left (when $\gamma(t)$ declines more rapidly than does $\alpha(t)$). For Swedish women, the background component clearly moved to the left between 1850 and 1950 and again between 1950 and 2000 because of a rapid decline in background mortality.

The background and senescent components combine to produce the overall pattern of $k(x,t)$, as shown in Figure B4 for Swedish women in 1950. In general, shifts over time of

Appendix Figure B2. Senescent Component of the Life-Table Aging Rate, Estimated With the Shifting Logistic Model: Swedish Women

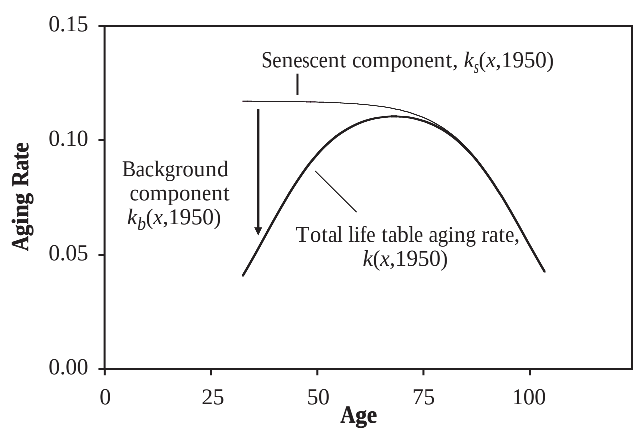


Appendix Figure B3. Background Component of the Life-Table Aging Rate, Estimated With the Shifting Logistic Model: Swedish Women

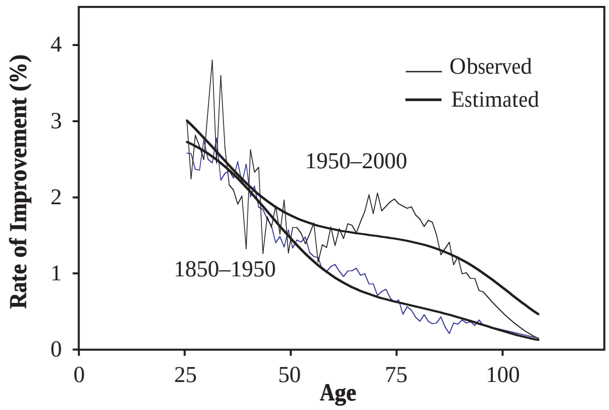


the ascending portion of the bell shape at lower ages are attributable to shifts in the background component $k_b(x,t)$, and shifts in the descending portion of the bell shape at higher ages are caused by shifts in the senescent component $k_s(x,t)$ (see the related discussion in Horiuchi and Wilmoth 1998). As a result, the overall bell-shaped pattern for $k(x,t)$ exhibits complex changes and can move to the left or right and become wider or narrower, depending on trends in $k_s(x,t)$ and $k_b(x,t)$, which, in turn, are determined by $\alpha(t)$, β, and $\gamma(t)$. This complexity makes it difficult to draw conclusions about trends in senescent mortality from the overall shape of $k(x,t)$. It is therefore preferable to analyze the background and

Appendix Figure B4. Senescent and Background Life-Table Aging Rate, Estimated With the Shifting Logistic Model: Swedish Women, 1950



Appendix Figure B5. Rate of Mortality Improvement, Observed and Estimated With the Shifting Logistic Model: Swedish Women



senescent components separately or to limit the analysis to the highest ages, when senescent mortality dominates.

Rate of Change Over Time

The relative derivative of the force of mortality with respect to time is defined as

$$\rho(x,t) = -\frac{1}{\mu(x,t)}\frac{\partial\mu(x,t)}{\partial t} \tag{B5}$$

and is called the rate of improvement in mortality (Keyfitz 1977; Vaupel 1986; Vaupel and Romo 2003).

Annual estimates for $\rho(x,t)$ tend to fluctuate widely, and the empirical analysis of this variable is therefore usually restricted to averages over periods of one or more decades. Figure B5 plots observed and model estimated values of $\rho(x,t)$ for Swedish women from 1850 to 1950 and from 1950 to 2000. To interpret these changes with age and over time, it is again useful to decompose $\rho(x,t)$ into two additive factors—the senescent component $\rho_s(x,t)$ and the background component $\rho_b(x,t)$:

$$\rho(x,t) = \rho_s(x,t) + \rho_b(x,t). \tag{B6}$$

The senescent component $\rho_s(x,t)$ is defined as the rate of improvement in mortality that would be observed in the absence of background mortality, and $\rho_b(x,t)$ equals the difference between $\rho(x,t)$ and $\rho_s(x,t)$.

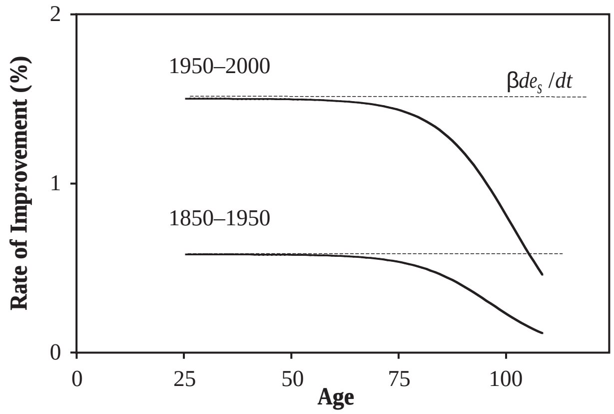
As shown in Appendix D, if the shifting assumption in Eq. (11) holds, then

$$\rho_s(x,t) = \dot{e}_s(t)k_s(x,t) \tag{B7}$$

$$\rho_b(x,t) = \dot{e}_s(t)k_b(x,t) - \frac{1}{\mu(x,t)}\frac{d\mu_b(t)}{dt}, \tag{B8}$$

where $\dot{e}_s(t)$ denotes the derivative of senescent life expectancy with respect to time: $\dot{e}_s(t) = de_s(t)/dt$. Note that Eqs. (B7) and (B8) are valid even if senescent mortality does

Appendix Figure B6. Senescent Component of the Rate of Mortality Improvement, Estimated the With Shifting Logistic Model: Swedish Women



not follow the logistic, provided that the shifting assumption holds. If the shifting logistic model does apply, substitution of Eq. (B3) in Eq. (B7) gives

$$\rho_s(x,t) = \dot{e}_s(t)\beta\left[1-\mu_s(x,t)\right], \tag{B9}$$

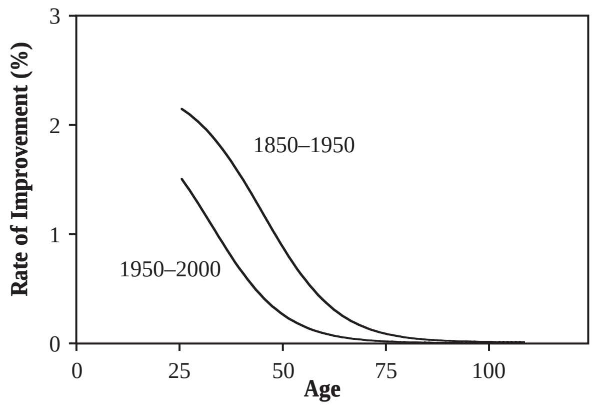
and $\gamma(t)$ can be substituted for $\mu_b(t)$ in Eq. (B8). Eq. (B9) is a more general version of the formula $\rho_s(t) = \dot{e}_s(t)\beta$ derived by Vaupel (1986) for the Gompertz model. (Note also that when background mortality is constant, $\rho(x,t) = \dot{e}_s(t)k(x,t)$.)

Figure B6 plots model estimates of $\rho_s(x,t)$ obtained from Eq. (B9) for Swedish women from 1850 to 1950 and from 1950 to 2000. The age pattern of $\rho_s(x,t)$ (but not its level) is the same as for $k_s(x,t)$: at the lowest ages, $\rho_s(x,t)$ is constant with age, equal to $\beta\dot{e}_s(t)$, because $\mu_s(x,t) \ll 1$. With advancing age, $\rho_s(x,t)$ declines and reaches 0 at very high ages, following the same pattern of relative decline as $k_s(x,t)$. The level of $\rho_s(x,t)$ is substantially higher for 1950–2000 than for 1850–1950 because senescent life expectancy rose at a more rapid pace in the former than in the latter period. The schedule for $\rho_s(x,t)$ shifts to the right as senescent life expectancy rises, as was the case for $k_s(x,t)$ and $\mu_s(x,t)$. Variations in the schedule $\rho_s(x,t)$ over time and with age are therefore the net result of two factors: (1) up or down movements over time that are due to variation in $\dot{e}_s(t)$ and (2) shifts to higher (lower) ages as $e_s(t)$ rises (falls).

Model estimates of the background component of the rate of improvement in mortality for Swedish women from 1850 to 1950 and from 1950 to 2000 are plotted in Figure B7. Over these two periods, the decline in background mortality was rapid, and the second term on the right-hand side of Eq. (B8) has dominated. This term is directly proportional to the rate of change in background mortality, and, since $\gamma(t)$ has declined over time (i.e., its derivative is negative), $\rho_b(x,t)$ has been positive, as is evident in Figure B7. The more rapidly $\gamma(t)$ declines, the more positive $\rho_b(x,t)$ becomes. In addition, $\rho_b(x,t)$ declines sharply with age and approaches zero at high ages. (Note that $\rho_b(x,t)$ is negative when background mortality is constant and senescent life expectancy is rising because $k_b(x,t)$ is negative.)

The senescent and background components combine to produce the patterns of change in the overall rate of improvement in mortality $\rho(x,t)$, as illustrated in Figure B8 for Swedish women for the period 1950–2000. At ages younger than about 70, the decline in $\rho(x,t)$

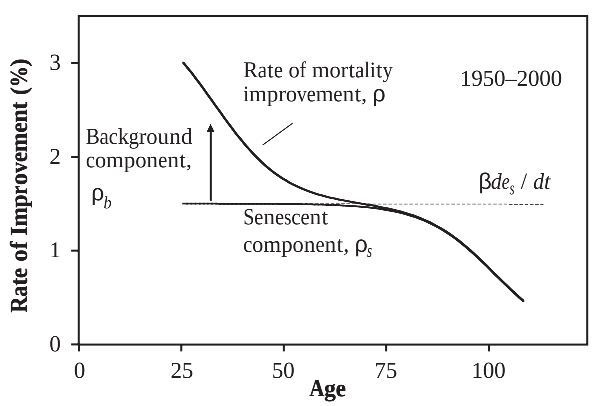
Appendix Figure B7. Background Component of the Rate of Mortality Improvement, Estimated With the Shifting Logistic Model: Swedish Women



with age is attributable to a decline in the background component, while the senescent component is approximately constant at $\beta\dot{e}_s(t)$. At ages older than 70, $\rho_b(x,t)$ is near zero and $\rho_s(x,t)$ declines, reaching zero at very old ages.

The pattern of $\rho(x,t)$ varies widely over time and among countries, as shown in Figures B9–B12, which compare model estimates of $\rho(x,t)$ for 1950–1960 and 1985–1995 for England and Wales, France, Italy, and Japan. To facilitate the interpretation of these results, the values of $\beta\dot{e}_s(t)$ for 1950–1960 and 1985–1995 are plotted as horizontal dashed lines. In the middle adult ages (about age 70) $\rho(x,t)$ is to this line. At younger ages, $\rho(x,t)$ is either

Appendix Figure B8. Decomposition of the Model Estimated Rate of Mortality Improvement: Swedish Women

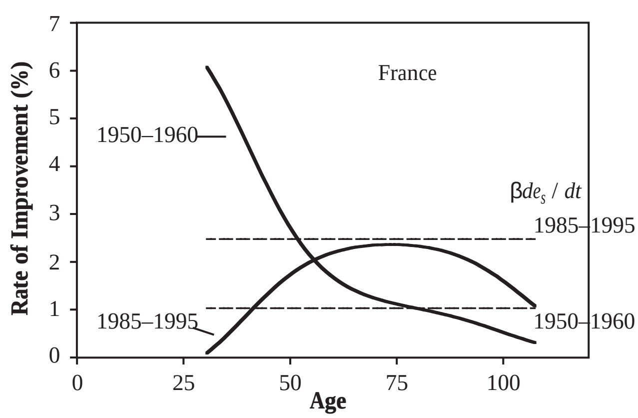


Appendix Figure B9. Rate of Mortality Improvement for Women in England and Wales, Estimated With the Shifting Logistic Model

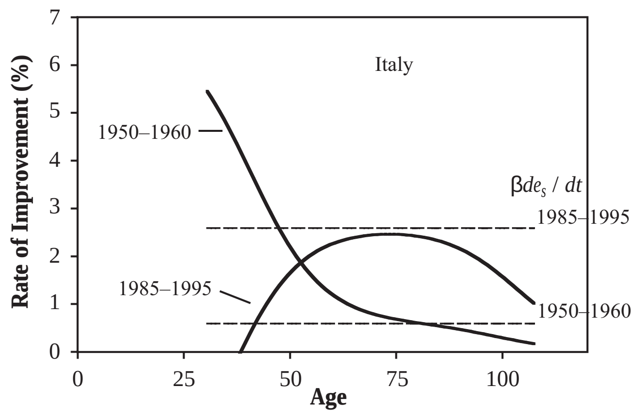
above (1950–1960) or below (1985–1995) this line, depending largely on the rate of decline in $\gamma(t)$. At older ages $\rho(x,t)$ declines with age and shifts to higher ages as senescent life expectancy rises.

Given the complexity of changes in $\rho(x,t)$, it is difficult to draw conclusions from them about overall trends in adult mortality. As was the case for $k(x,t)$, it is preferable to analyze the background and senescent components of $\rho(x,t)$ separately. Limiting the analysis to the highest ages, where the senescent component dominates, is somewhat helpful, but it is difficult to determine whether changes at the highest ages are due

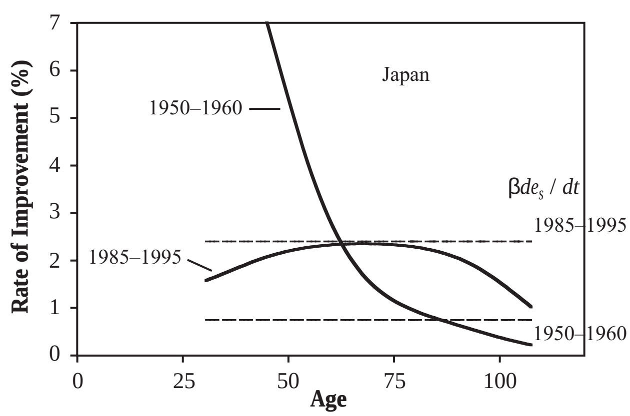
Appendix Figure B10. Rate of Mortality Improvement for Women in France, Estimated With the Shifting Logistic Model



Appendix Figure B11. Rate of Mortality Improvement for Women in Italy, Estimated With the Shifting Logistic Model



Appendix Figure B12. Rate of Mortality Improvement for Women in Japan, Estimated With the Shifting Logistic Model



to shifting (caused by a change in the level of senescent life expectancy) or to an up or down movement (caused by variation in the rate of change in senescent life expectancy).

This analysis of the rate of change in the force of mortality leads to two conclusions. First, the age pattern of $\rho(x,t)$ has changed substantially in recent decades in many countries. This change makes it likely that the rate of improvement in mortality will not be constant in the future, as assumed in some existing projection methods. Second, the factors that are responsible for the variation in $\rho(x,t)$ include different trends in background and senescent mortality and a shifting pattern of senescent mortality.

APPENDIX C: DECOMPOSITION OF THE AGING RATE FOR THE SHIFTING LOGISTIC MODEL

The objective is to derive Eqs. (B3) and (B4). The first step is to find an equation relating $k(x,t)$ to the parameters in the shifting logistic model. Substitution of Eq. (6) in Eq. (B1) yields

$$k(x,t) = \frac{\dfrac{\partial \mu(x,t)}{\partial x}}{\mu(x,t)}$$

$$= \frac{\dfrac{\partial}{\partial x}\left[\dfrac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta xt}} + \gamma(t)\right]}{\left[\dfrac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta xt}} + \gamma(t)\right]}$$

$$= \frac{\dfrac{\beta\alpha(t)e^{\beta x}}{\left[1+\alpha(t)e^{\beta x}\right]^2}}{\left[\dfrac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta xt}} + \gamma(t)\right]}$$

$$= \frac{\beta\alpha(t)e^{\beta x}}{\left[1+\alpha(t)e^{\beta x}\right]\left[\alpha(t)e^{\beta x} + \gamma(t)(1+\alpha(t)e^{\beta x})\right]}. \tag{C1}$$

The senescent component $k_b(x,t)$ of $k(x,t)$ is defined as the aging rate that would be observed in the absence of background mortality. Substitution of $\mu_b(x,t) = \gamma(t) = 0$ in Eq. (C1) gives

$$k_s(x,t) = \frac{\beta}{1+\alpha(t)e^{\beta x}}$$

$$= \beta\left[1-\mu_s(x,t)\right] \tag{C2}$$

thus confirming Eq. (B3).

The background component $k_s(x,t)$ of $k(x,t)$ is defined as the difference between $k(x,t)$ and $k_s(x,t)$:

$$k_b(x,t) = k(x,t) - k_s(x,t). \tag{C3}$$

Substitution of Eqs. (C1) and (C2) in Eq. (C3) yields

$$k_b(x,t) = \frac{\beta\alpha(t)e^{\beta x}}{\left[1+\alpha(t)e^{\beta x}\right]\left[\alpha(t)e^{\beta x} + \gamma(t)(1+\alpha(t)e^{\beta x})\right]} - \frac{\beta}{1+\alpha(t)e^{\beta x}}. \tag{C4}$$

Simplification of Eq. (C4) gives Eq. (B4).

APPENDIX D: DECOMPOSITION OF THE RATE OF MORTALITY IMPROVEMENT

The aim of this appendix is to derive Eqs. (B7) and (B8) for the senescent and background components of the rate of mortality improvement, provided that the shifting assumption in Eq. (11) holds.

Senescent Component $\rho_s(x,t)$

By definition, the senescent component equals the rate of mortality observed when background mortality equals zero, so that

$$\rho_s(x,t) = -\frac{1}{\mu_s(x,t)}\frac{\partial \mu_s(x,t)}{\partial t} = -\frac{\partial \ln \mu_s(x,t)}{\partial t}. \tag{D1}$$

To derive Eq. (B7) from Eq. (D1), it is necessary first to examine the relationship between $k_s(x,t)$ and $\mu_s(x,t)$ in more detail. The relative derivative of $\mu_s(x,t)$ with respect to age is defined as

$$k_s(x,t) = \frac{1}{\mu_s(x,t)}\frac{\partial \mu_s(x,t)}{\partial x} \tag{D2}$$

so that

$$\mu_s(x,t) = \mu_s(0,t)\exp\left[\int_0^x k_s(a,t)da\right]. \tag{D3}$$

If the shifting assumption in Eq. (11) holds, then changes in $k_s(x,t)$ occur through the same shifts to higher or lower ages, as in $\mu_s(x,t)$:

$$k_s(x,t) = k_s(x - S(t),t_0), \tag{D4}$$

where S equals the amount of the shift in years up or down the age axis $\mu_s(x,t)$ or $k_s(x,t)$ between t and t_0. When senescent life expectancy is rising, $S(t)$ is positive, and Eq. (D3) holds for $x > S(t)$ with $k_s(x,t)$ for $x < S(t)$; when S is negative, Eq. (D3) holds for $x > 0$). The shift in S is a function of t and t_0, but subscripts will be dropped to simplify the notation. In most populations, it is possible to select the base year t_0 so that S is positive because senescent life expectancy has risen between t and t_0. The following derivation assumes that this is the case.

With $S(t) > 0$, $\mu_s(S(t),t) = \mu_s(0,t_0)$ and substitution of this and of Eq. (D4) in Eq. (D3) gives

$$\mu_s(x,t) = \mu_s(0,t_0)\exp\left\{\int_{S(t)}^x k_s(a - S(t),t_0)da\right\} \tag{D5}$$

for $x > S(t)$ and $\mu_s(x,t) = 0$ otherwise.

Substitution of Eq. (D5) in Eq. (D1) now gives

$$\rho_s(x,t) = -\frac{\partial \ln\left[\mu_s(0,t_0)e^{-\int_S^x k_s(a-S(t),t_0)da}\right]}{\partial t}$$

$$= -\frac{\partial}{\partial t}\int_{S(t)}^{x} k_s(a - S(t), t_0)da$$

$$= -\frac{\partial}{\partial t}\int_{0}^{a-S(t)} k_s(y, t_0)dy$$

$$= \frac{dS(t)}{dt} k_s(x - S(t), t_0). \tag{D6}$$

And substitution of Eqs. (D4) and (10) in Eq. (D6) yields

$$\rho_s(x,t) = \frac{dS(t)}{dt} k_s(x,t)$$

$$= \frac{de_s(t)}{dt} k_s(x,t), \tag{D7}$$

thus confirming Eq. (B7).

Background Component $\rho_b(x,t)$

The background component of the rate of mortality improvement is defined as

$$\rho_b(x,t) = \rho(x,t) - \rho_s(x,t). \tag{D8}$$

Substitution of

$$\rho(x,t) = -\frac{1}{\mu(x,t)}\frac{\partial\mu_s(x,t)}{\partial t} - \frac{1}{\mu(x,t)}\frac{\partial\mu_b(x,t)}{\partial t} \tag{D9}$$

and of Eq. (D7) in Eq. (D8) gives

$$\rho_b(x,t) = -\frac{1}{\mu(x,t)}\frac{\partial\mu_s(x,t)}{\partial t} - \frac{1}{\mu(x,t)}\frac{\partial\mu_b(x,t)}{\partial t} - \frac{de_s(t)}{dt}k_s(x,t)$$

$$= \frac{\mu_s(x,t)}{\mu(x,t)}\frac{de_s(t)}{dt}k_s(x,t) - \frac{1}{\mu(x,t)}\frac{\partial\mu_b(x,t)}{\partial t} - \frac{de_s(t)}{dt}k_s(x,t)$$

$$= \left[\frac{\mu_s(x,t)}{\mu(x,t)} - 1\right]\frac{de_s(t)}{dt}k_s(x,t) - \frac{1}{\mu(x,t)}\frac{\partial\mu_b(x,t)}{\partial t}. \tag{D10}$$

The first term on the right-hand side of Eq. (D10) can be simplified by noting that $d\mu_b(x,t) / dx = 0$ because background mortality does not vary with age. This implies that

$$\frac{\partial\mu_s(x,t)}{\partial x} = \frac{\partial\mu(x,t)}{\partial x} \tag{D11}$$

and therefore

$$\frac{k(x,t)}{k_s(x,t)} = \frac{\mu_s(x,t)[\partial\mu(x,t)/dx]}{\mu(x,t)[\partial\mu_s(x,t)/dx]} = \frac{\mu_s(x,t)}{\mu(x,t)}. \tag{D12}$$

Substitution of Eq. (D12) in Eq. (D10) with $k(x,t) = k_s(x,t) + k_b(x,t)$ gives

$$\rho_b(x,t) = \left[\frac{k(x,t)}{k_s(x,t)} - 1\right]\frac{de_s(t)}{dt}k_s(x,t) - \frac{1}{\mu(x,t)}\frac{\partial\mu_b(x,t)}{\partial t}$$

$$= k_b(x,t)\frac{de_s(t)}{dt} - \frac{1}{\mu(x,t)}\frac{\partial\mu_b(x,t)}{\partial t}, \tag{D13}$$

thus confirming Eq. (B8).

A simpler expression for $\rho(x,t)$ can be obtained if background mortality is constant, as appears to be approximately the case over the past two decades in the 14 countries plotted in Figure 3c. With $d\mu_b(t)(x,t) / dt = 0$, the second term on the right side of Eq. (D13) disappears. The sum of the senescent and background components then becomes

$$\rho(x,t) = \frac{de_s(t)}{dt}k_s(x,t) + \frac{de_s(t)}{dt}k_b(x,t) = \frac{de_s(t)}{dt}k(x,t). \tag{D14}$$

In this special case, the age pattern of $\rho(x,t)$ has the same shape as $k(x,t)$, and the entire schedule of $\rho(x,t)$ is proportional to the rate of change in senescent life expectancy. The three schedules—$\mu(x,t)$, $k(x,t)$, and $\rho(x,t)$—maintain their shape over time and shift to higher or lower ages at the same pace as senescent life expectancy rises or falls.

REFERENCES

Beard, R.E. 1971. "Some Aspects of Theories of Mortality, Cause of Death Analysis, Forecasting and Stochastic Processes." Pp. 57–68 in *Biological Aspects of Demography,* edited by W. Brass. New York: Barnes & Noble.

Bongaarts, J. and G. Feeney. 2002. "How Long Do We Live?" *Population and Development Review* 28(1):13–29.

———. 2003. "Estimating Mean Lifetime." *Proceedings of the National Academy of Sciences US* 100(23):13127–33.

Booth, H., J. Maindonald, and L. Smith. 2002. "Applying Lee-Carter Under Conditions of Variable Mortality Decline." *Population Studies* 56:325–36.

Carter, R. and A. Prskawetz. 2001. "Examining Structural Shifts in Mortality Using the Lee-Carter Method." Working Paper 2001-007. Max Planck Institute for Demographic Research, Germany.

Gavrilov, L.A. and N.S. Gavrilova. 1991. *The Biology of Life Span: A Quantitative Approach,* edited by V.P. Skulachev. Chur, Switzerland: Harwood Academic.

Gompertz, B. 1825. "On the Nature of the Function Expressive of the Law of Mortality." *Philosophical Transactions* 27:513–85.

Himes, C.L., S.H. Preston, and G.A. Condran. 1994. "A Relational Model of Mortality at Older Ages in Low Mortality Countries." *Population Studies* 48:269–91.

Hollman, F.W., T.J. Mulder, and J.E. Kallan. 2000. "Methodology and Assumptions for the Population Projections of the United States: 1999 to 2100." Working Paper 38. Population Division, U.S. Census Bureau, Washington, DC.

Horiuchi, S. and A.J. Coale. 1990. "Age Patterns of Mortality for Older Women: An Analysis Using the Age-Specific Rate of Mortality Change With Age." *Mathematical Population Studies* 2:245–67.

Horiuchi, S. and J.R. Wilmoth. 1998. "Deceleration in the Age Pattern of Mortality at Older Ages." *Demography* 35:391–412.

Kannisto, V. 1996. *The Advancing Frontier of Survival: Life Tables for Old Age*. Odense, Denmark: Odense University Press.

Keyfitz, N. 1977. *Applied Mathematical Demography*. New York: John Wiley and Sons.

———. 1984. "Choice of Function for Mortality Analysis: Effective Forecasting Depends on a Minimum Parameter Representation." Pp. 225–43 in *Methodologies for the Collection and Analysis of Mortality Data,* edited by J. Vallin, J.H. Pollard, and L. Heligman. Liege, Belgium: Ordina Editions.

———. 1991. "Experiments in the Projection of Mortality." *Canadian Studies in Population* 18(2):1–17.

Lee, R. 1998. "Probabilistic Approaches to Population Forecasting." In *Frontiers of Population Forecasting,* edited by W. Lutz, J.W. Vaupel, and D.A. Ahlburg. Supplement to *Population and Development Review* 24:156–90.

———. 2000. "The Lee-Carter Method for Forecasting Mortality, With Various Extensions and Applications." *North American Actuarial Journal* 4(1):80–93.

Lee, R.D. and L.R. Carter. 1992. "Modeling and Forecasting U.S. Mortality." *Journal of the American Statistical Association* 87(419):659–71.

Lee, R. and T. Miller. 2001. "Evaluating the Performance of the Lee-Carter Method for Forecasting Mortality." *Demography* 38:537–49.

Makeham, W. 1860. "On the Law of Mortality and the Construction of Annuity Tables." *Journal of the Institute of Actuaries* 13:325–58.

McNown, R. 1992. "Comment" on Lee, Ronald D. and Lawrence R. Carter, "Modeling and Forecasting U.S. Mortality." *Journal of the American Statistical Association* 87(419):659–72.

Olshansky, S.J. 1988. "On Forecasting Mortality." *Milbank Quarterly* 66:482–530.

Perks, W. 1932. "On Some Experiments in the Graduation of Mortality Statistics." *Journal of the Institute of Actuaries* 63:12–57.

Pollard, J.H. 1987. "Projection of Age-Specific Mortality Rates." Pp. 55–69 in *Population Bulletin of the United Nations*. Nos. 21–22. New York: United Nations.

Tabeau, E., A. Jeths, and C. Heathcote. 2001. *Forecasting Mortality in Developed Countries.* Dordrecht, The Netherlands: Kluwer Academic.

Technical Panel on Assumptions and Methods. 1999. Report to the Social Security Advisory Board. Available on-line at http://www.ssab.gov/Rpt99.pdf

Thatcher, A.R. 1999. "The Long-Term Pattern of Adult Mortality and the Highest Attained Age." *Journal of the Royal Statistical Society* 162 Part 1:5–43.

Thatcher A.R., V. Kannisto, and J.W. Vaupel. 1998. *The Force of Mortality at Ages 80 to 120*. Odense, Denmark: Odense University Press.

Tuljapurkar, S., N. Li, and C. Boe. 2000. "A Universal Pattern of Mortality Decline in the G7 Countries." Letter. *Nature* 405:789–92.

Vaupel, J.W. 1986. "How Change in Age-Specific Mortality Affects Life Expectancy." *Population Studies* 40(1):147–57.

Vaupel, J.W. and V.C. Romo. 2003. "Decomposing Change in Life Expectancy: A Bouquet of Formulas in Honor of Nathan Keyfitz's 90th Birthday." *Demography* 40:201–16.

United Nations. 2002. *World Population Prospects 2002.* New York: United Nations.

———. 2003. *Long-Range Population Projections: Proceedings of the United Nations Technical Working Group on Long-Range Population Projections*. New York: United Nations Population Division. Available on-line at http://www.un.org/esa/population/publications/longrange/long-range_working-paper_final.PDF